ON THE MATHEMATICAL MODEL OF ROTATING SHAFT

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Abstract: *The paper deals with a shaft subjected to an axial pressure with force P and relatively high moment of rotation M. In order to determne the lines of deflection we propose various approaches. The assessment of critical values of P and M is also discussed when loss of stability is available. Thus two fourth-order eigenvalue problems are considered and their mixed variational models are proposed. Finally, some numerical experimental results are presented.*

Keywords: ROTATING SHAFT, EIGENVALUE PROBLEM, VARIATIONAL FORMULATION, AXIAL PRESSURE

1. Introduction and Problem Setting

The mathematical models are an essential tool for investigation the stability and measurement of many mechanical structures. The corresponding differential equations are usually of second or fourth order [1]. In reference to determination of the important critical values, one is interested in estimation of the first eigenpair of the differential spectral problem (see e.g. [2]). The question is not only to offer an adequate mathematical model, but also to give sufficiently precise numerical solution that is convenient for computer implementation. Variational numerical methods and finite element methods especially are commonly used in engineering studies [6].

In this paper, we consider a rotating shaft with length l under the influence of one-sided axial pressure force P and moment of rotation M (see Fig. 1).

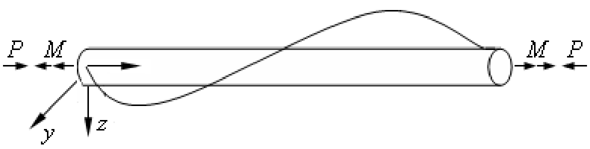

Fig. 1 *Rotating shaft under the influence of axial pressure force P and rotating moment M.*

The shaft deflection curve is situated in *Oyz*-plane, where $y = y(x)$, $z = z(x)$ and $x\in[0,l]$. First, we consider the shaft in a slightly bent state. Then, the moments of deflection Pz and Py are small and in the beginning there is only a torsion.

Considering a relatively high rotating moment M, the combination of this moment and the axial force P could cause screw shaped axial deformation. So, we are interested in the critical values of P and M.

The moment M is decomposed by My' and Mz' in the corresponding axial directions (Fig. 1). Then the equations of deflection are (see [2,3]):

$$\left|\begin{aligned}&-\alpha z''= P z - M y'\\&-\alpha y''= P y + M z',\end{aligned}\right. \qquad (1)$$

where $\alpha=\alpha(x)$ is the flexural rigidity. Here, we adopt $\alpha_x=\alpha_y$ and hereinafter we suppose that $\alpha=const$.

The system (1) has homogeneous boundary conditions:

$$y(0)=z(0)=y(l)=z(l)=0.$$

2. Mathematical Fourth-order Model

To start with, we will determine the displacements in z-direction. For this purpose, we eliminate $y(x)$ from (1). We differentiate the first equation of (1) and using the second one, it follows that:

$$\begin{aligned}-\alpha z''' &= P z' - M y''\\ &= P z' + M\left(\frac{P}{\alpha}y+\frac{M}{\alpha}z''\right),\end{aligned}$$

or

$$-\alpha z''' = P z' + \frac{MP}{\alpha}y + \frac{M^2}{\alpha}z''. \qquad (2)$$

Multiplying the last equation by α and differentiating again, we ontain:

$$-\alpha^2 z^{IV} = \alpha P z'' + M^2 z'' + M P y'.$$

Having in mind that

$$M y' = \alpha z'' + P z,$$

we easily get

$$z^{IV} + \frac{2P}{\alpha}z'' + \frac{M^2}{\alpha^2}z'' + \frac{P^2}{\alpha^2}z = 0. \qquad (3)$$

In order to determine the boundary conditions of (3), it has to use (2) with $y = 0$. Thus:

$$\begin{aligned}&z(0)=z(l)=0;\\&\left[\alpha z''' + P z' + \frac{M^2}{\alpha}z'\right]_{\substack{x=0\\x=l}} = 0.\end{aligned} \qquad (4)$$

Case 1. Let the moment M be given. Then $P=\lambda$ plays the part of the unknown eigenvalue.

The eigenvalue problem (3) in this case is nonlinear with respect to λ.

Case 2. The force P is given and $M^2=\lambda$. Then we obtain linear fourth-order elliptic problem with self-adjoint operator.

It should be noted that in both cases the eigenvalue parameter λ appears in the boundary conditions (4).

Remark 1. *Variational aspects of one-dimensional fourth-order problems with eigenvalue parameter which appear linearly in the boundary conditions are considered in [5] (see also [4]).*

Remark 2. *It could be mentioned the particular case when $P=0$, i.e. there is no axial force P. Then ($M^2=\lambda$):*

$$-\alpha^2 z^{IV} = \lambda z''$$

and we have a self-adjoint differential form.

3. Variational Formulations

We use the Sobolev space of order *s*: $H^s(0,l),\ s=1;2$. Then the variational functional set *V* is defined by:

$$V = H^2(0,l) \cap H_o^1(0,l).$$

Let $z_1(x)\in V$, then $z_1(0)=z_1(l)=0$. Integration by parts reveals that:

$$\int_0^l z^{IV} z_1\, dx = \int_0^l z'' z_1''\, dx - z''(l) z_1'(l) + z''(0) z_1'(0).$$

Moreover,

$$\int_0^l z'' z_1\, dx = -\int_0^l z' z_1'\, dx.$$

We introduce the following bilinear forms for any $u,v\in V$:

$$a(u,v) = \int_0^l u'' v''\, dx - u''(l)v'(l) + u''(0)v'(0);$$

$$b_1(u,v) = \frac{2}{\alpha}\int_0^l u' v'\, dx; \quad b_2(u,v) = \frac{1}{\alpha^2}\int_0^l u' v'\, dx;$$

$$c(u,v) = \frac{1}{\alpha^2}\int_0^l u\, v\, dx.$$

Here, the bilinear form $a(\cdot\,,\cdot)$ is not symmetric.

From (3), (4) we easily obtain the two variational forms:

Case 1. $P=\lambda$ and *M* is known

$$a(z,z_1) - M^2 b_2(z,z_1) = \lambda b_1(z,z_1) - \lambda^2 c(z,z_1) \quad \forall z_1\in V. \tag{5}$$

Case 2. $M^2=\lambda$ and *P* is known

$$a(z,z_1) - P b_1(z,z_1) + P^2 c(z,z_1) = \lambda b_2(z,z_1) \quad \forall z_1\in V. \tag{6}$$

Here it should be listed some disadvantages concerning fourth-order variational models (5) and (6). Namely:

- Both eigenvalue problems are non-symmetric;
- The eigenvalue parameter appears in the boundary conditions (4);
- In order to determine critical values of the force *P*, we have to solve quadratic (non-linear) problem (5), which means that we have to undertake some linearization procedure;
- In the finite element method C^1 – requirement for smoothness should be satisfied [6]. So that, the approximation polynomials are of degree $n\ge 3$.

Based on all this, we propose numerical solution to be found using system (1) subject to homogeneous boundary conditions

$$y(0) = z(0) = y(l) = z(l) = 0.$$

For this purpose we will obtain the corresponding variational formulations of the problem (1) in both cases under consideration.

Multiplying the equations (1) by functions $y_1(x)\in H_0^1(0,l)$ and $z_1(x)\in H_0^1(0,l)$, respectively, and integrating by parts, we get:

$$\left|\begin{aligned} &\alpha\int_0^l z' z_1'\, dx = P\int_0^l z z_1\, dx - M\int_0^l y' z_1\, dx \\ &\alpha\int_0^l y' y_1'\, dx = P\int_0^l y y_1\, dx - M\int_0^l y_1' z\, dx. \end{aligned}\right. \tag{7}$$

Subtracting the equations form (7), we obtain the following variational equation:

$$\alpha\int_0^l z' z_1'\, dx + \alpha\int_0^l y' y_1'\, dx = P\int_0^l z z_1\, dx + P\int_0^l y y_1\, dx - M\int_0^l y' z_1\, dx - M\int_0^l y_1' z\, dx, \tag{8}$$

which is symmetric with respect to the couples of functions $\big(z(x),y(x)\big)$ and $\big(z_1(x),y_1(x)\big)$, which belong to the space $H_0^1(0,l)\times H_0^1(0,l)$.

From (8) we come to variational models of (1) subject to homogeneous boundary conditions as follows.

Case 1. $P=\lambda$ and *M* is known

Then we obtain the following variational problem corresponding to (1): Find a number $\lambda\in R$ and a couple of functions $\big(z(x),y(x)\big)\in H_0^1(0,l)\times H_0^1(0,l)$ such that:

$$\tilde{a}(z,z_1) + \tilde{a}(y,y_1) + \tilde{a}_1(y,z_1) + \tilde{a}_1(y_1,z) = \lambda\left(\tilde{b}(z,z_1) + \tilde{b}(y,y_1)\right), \quad \forall\,(z_1,y_1) \in H_0^1(0,l)\times H_0^1(0,l), \tag{9}$$

where the bilinear forms $\tilde{a}(\cdot\,,\cdot)$, $\tilde{a}_1(\cdot\,,\cdot)$ and $\tilde{b}(\cdot\,,\cdot)$ are defined by

$$\tilde{a}(u,v) = \alpha\int_0^l u'v'dx; \quad \tilde{a}_1(u,v) = M\int_0^l u'v\,dx, \quad \forall (u,v) \in H_0^1(0,l)\times H_0^1(0,l)$$

and

$$\tilde{b}(u,v) = \int_0^l u\,v\,dx, \quad \forall (u,v) \in L_2(0,l)\times L_2(0,l).$$

Let us underline that the variational model (9) is symmetric with respect to the couples of functions $(z(x),y(x))$ and $(z_1(x),y_1(x))$, belonging to the space $H_0^1(0,l)\times H_0^1(0,l)$. At that in (9) the eigenvalue parameter λ appears linearly, which is the main advantage of the proposed model.

Case 2. *P* is known

Taking into account that, in contrast to the variational model (6), corresponding to fourth-order problem (3), this time M appears just linearly in (8), we will denote $\lambda = M$ instead of $\lambda = M^2$.

Then the variational problem corresponding to (1) in this case is: Find a number $\lambda \in R$ and a couple of functions $(z(x),y(x)) \in H_0^1(0,l)\times H_0^1(0,l)$ such that:

$$\tilde{a}(z,z_1) + \tilde{a}(y,y_1) - \tilde{b}_1(z,z_1) - \tilde{b}_1(y,y_1) = \lambda\left(\tilde{a}_2(y,z_1) + \tilde{a}_2(y_1,z)\right), \quad \forall\,(z_1,y_1) \in H_0^1(0,l)\times H_0^1(0,l), \tag{10}$$

where the bilinear forms $\tilde{a}(\cdot\,,\cdot)$, $\tilde{a}_2(\cdot\,,\cdot)$ and $\tilde{b}_1(\cdot\,,\cdot)$ are defined by

$$\tilde{a}(u,v) = \alpha\int_0^l u'v'dx; \quad \tilde{a}_2(u,v) = -\int_0^l u'v\,dx, \quad \forall (u,v) \in H_0^1(0,l)\times H_0^1(0,l)$$

and

$$\tilde{b}_1(u,v) = P\int_0^l u\,v\,dx, \quad \forall (u,v) \in L_2(0,l)\times L_2(0,l).$$

As well as in the previous case, the variational model (10) which we obtain is symmetric with respect to the couples of functions $(z(x),y(x))$ and $(z_1(x),y_1(x))$, belonging to the space $H_0^1(0,l)\times H_0^1(0,l)$.

4. Discretization and Matrix Representations

We will solve numerically the problems (9) and (10) by means of finite element method, so that first we should discretize these problems.

Let $V_h \subset H_0^1(0,l)$ be a finite element space.

The discrete variational problem corresponding to (9) reads: Find a number $\lambda_h \in R$ and a couple of functions $(z_h(x),y_h(x)) \in V_h\times V_h$ such that:

$$\tilde{a}(z_h,z_{1,h}) + \tilde{a}(y_h,y_{1,h}) + \tilde{a}_1(y_h,z_{1,h}) + \tilde{a}_1(y_1,z_{1,h}) = \lambda\left(\tilde{b}(z_h,z_{1,h}) + \tilde{b}(y_h,y_{1,h})\right), \quad \forall\,(z_{1,h},y_{1,h}) \in V_h\times V_h. \tag{11}$$

Similarly, the discrete variational problem corresponding to (10) reads: Find a number $\lambda_h \in R$ and a couple of functions $(z_h(x),y_h(x)) \in V_h\times V_h$ such that:

$$\tilde{a}(z_h,z_{1,h}) + \tilde{a}(y_h,y_{1,h}) - \tilde{b}_1(z_h,z_{1,h}) - \tilde{b}_1(y_h,y_{1,h}) = \lambda\left(\tilde{a}_2(y_h,z_{1,h}) + \tilde{a}_2(y_{1,h},z_h)\right), \quad \forall\,(z_{1,h},y_{1,h}) \in V_h\times V_h. \tag{12}$$

Let the nodes of the finite element partition be $a_j,\ j=1,2,\ldots,N$ and their corresponding shape functions be $\varphi_j(x),\ j=1,2,\ldots,N$.

Let us also denote $Z=(z_h(a_1), z_h(a_2),\ldots, z_h(a_N))^T$ and $Y=(y_h(a_1), y_h(a_2),\ldots, y_h(a_N))^T$.

Then the discrete variational problem (11) takes the following matrix representation:

$$\begin{pmatrix} A & A_1 \\ A_1^T & A \end{pmatrix}\begin{pmatrix} Z \\ Y \end{pmatrix} = \lambda_h\begin{pmatrix} B & O \\ O & B \end{pmatrix}\begin{pmatrix} Z \\ Y \end{pmatrix}, \tag{13}$$

where

$$A=\left(\alpha\int_0^l \varphi_i'\,\varphi_j'\,dx\right)_{i,j=1}^N;\ A_1=\left(M\int_0^l \varphi_i\,\varphi_j'\,dx\right)_{i,j=1}^N;\ B=\left(\int_0^l \varphi_i\,\varphi_j\,dx\right)_{i,j=1}^N$$

and *O* is the null $N\times N$ – matrix.

Similarly, the matrix representation of the discrete variational problem (12) is:

$$\begin{pmatrix} A-B_1 & O \\ O & A-B_1 \end{pmatrix}\begin{pmatrix} Z \\ Y \end{pmatrix} = \lambda_h\begin{pmatrix} O & A_2^T \\ A_2^T & O \end{pmatrix}\begin{pmatrix} Z \\ Y \end{pmatrix},$$

where

$$A=\left(\alpha\int_0^l \varphi_i'\,\varphi_j'\,dx\right)_{i,j=1}^N;\ A_2=\left(-\int_0^l \varphi_i\,\varphi_j'\,dx\right)_{i,j=1}^N;\ B=\left(P\int_0^l \varphi_i\,\varphi_j\,dx\right)_{i,j=1}^N$$

and *O* is the null $N\times N$ – matrix.

5. Numerical Results

In this section we give results from numerical experiments which verify and approve our theoretical results and propositions.

We will demonstrate the proposed method for Case 1. Let us put $M=0$ into the equation (3) (or, equivalently, into the system (1)). Thus, taking into account the boundary conditions, we get that the exact values for the eigenvalue parameter $\lambda = P$ are equal to $\frac{k^2\pi^2}{l^2}, k=1,2,3,...$

The selected choice $M=0$ hardly make a realistic example, but it serves as an confirmation of the proposed method and an illustration how does the method works; at that the exact eigenvalues are known.

Our numerical finite element implementation is done for $l=\pi$. We divide uniformly the interval $(0,\pi)$ into N subintervals, N = 10; 20; 30; 40; 50; 60, so that the mesh parameter h is equal to $\frac{\pi}{N}$. We solve the variational discrete problem (11) using linear finite elements for V_h. It is also possible to use more refined finite elements (quadratic, cubic, etc.), but here we would like to demonstrate the proposed method by means of the simplest finite elements.

In Table 1 approximations $\lambda_{i,h}, i=1,2,3,4$ of the first four exact eigenvalues $\lambda_i, i=1,2,3,4$ by means of the proposed method solving are presented. The relative error $\frac{|\lambda_{i,h}-\lambda_i|}{\lambda_i}$ is also given. As it is expected, the approximate values are greater than the exact ones, because of the fact that conforming finite element method is used.

Table 1: *Approximations of the exact eigenvalues obtained after finite element implementation of the proposed method by means of linear finite elements*

	$\lambda_{1,h}$ Error, %	$\lambda_{2,h}$ Error, %	$\lambda_{3,h}$ Error, %	$\lambda_{4,h}$ Error, %
N = 10	1.008251453 0.83	4.133251787 3.33	9.683821975 7.6	18.19247316 13.7
N = 20	1.002057854 0.21	4.033005812 0.82	9.167753489 1.86	16.53300815 3.33
N = 30	1.000914186 0.09	4.014642920 0.37	9.074263077 0.83	16.23529772 1.47
N = 40	1.000514148 0.05	4.008231418 0.21	9.041714009 0.46	16.13204035 0.83
N = 50	1.000329030 0.03	4.005266555 0.13	9.026679377 0.3	16.08422401 0.53
N = 60	1.000228484 0.02	4.003656744 0.09	9.018520713 0.21	16.05883991 0.37
Exact eigenvalues	1	4	9	16

6. Conclusions

The mathematical model of rotating shaft could be analyzed by two approaches: using fourth-order two-point equation (3), (4) or a system of two second-order equations (1) with homogeneous boundary conditions.

Loss of stability of the shaft is related to critical values of the pressure P and rotational moment M. They can be determined by the first eigenvalues of corresponding eigenvalue problems.

This paper confirms that the use of mixed-type variational presentation is preferable. Then, the finite element method is easier for numerical implementation. Such being the case, it works with symmetric bilinear forms and, consequently, with symmetric resulting matrices. At that, the appearance of the eigenvalue parameter into the boundary conditions is avoided and also eigenvalue parameter appears just linearly into the proposed mixed-type mathematical model.

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