A COMPARISON OF SEQUENTIAL QUALITY CONTROL METHODS

Prof. Gurevich G., Mrs. Zohar L.
Department of Industrial Engineering and Management – SCE-Shamoon College of Engineering, Ashdod, Israel

gregoryg@sce.ac.il, limorzohar1986@gmail.com

Abstract: *This research considers the problem of sequential quality control and presents different methods for the quick detection, with low false alarm rate, of a change in a stochastic system. The paper focuses on recently proposed control schemes called Nested Plans. These schemes include two unknown parameters that essentially impact their efficiency. The research presents a way to find the optimal values of the parameters, and shows that Nested Plans with the correct choice of the parameters are very efficient, robust and simple for practical applications.*

Keywords: QUALITY CONTROL, SEQUENTIAL ANALYSIS, NESTED PLANS, CUSUM.

1. Introduction

There are extensive references in statistics and engineering literature on the subject of early detection, with low false alarm rate, of parameter changes in stochastic systems. Such problems are very important in the context of quality and reliability control. The ingredients of the change-point detection problem are a sequence of observations whose baseline distribution has some density that may change to an alternative density. A common performance measure for any inspection scheme is the average run length (ARL). Let N be a stopping time (the random variable corresponding to the time when an alarm is raised). The in-control ARL (average run length until a false alarm) and the out-of-control ARL (average run length from change to its detection) are defined as expectations of the stopping time N under the pre-change distribution and the post-change distribution, respectively. The first formal sequential method has been proposed by Shewhart (Shewhart [1]). He proposed raising an alarm the first time that an observation exceeds the known baseline mean by more than three standard deviations. The method is known to be very good in detecting a large change quickly. Later, various more efficient methods have been proposed. Lorden [2] proved that the minimum over all stopping times with in-control ARL$\geq A$ is

$$\frac{\left(1+o(1)\right) log\ A}{E_{F_2}\left(log\left(f_2(X)/f_1(X)\right)\right)}\ , \qquad (1)$$

where $E_{F_2}\left(X\right)$ denotes the expectation of the random variable X under the assumption that X is distributed according to the post-change distribution, $f_1\left(x\right)$ and $f_2\left(x\right)$ are the pre-change and the post-change densities, respectively, $o\left(1\right)\to 0$ as $A\to\infty$. Relatively recent results showed that if the pre-change and the post-change distributions are known and $A\to\infty$, then out-of-control ARL of the CUSUM and Shiryaev-Roberts control charts achieve the asymptotic lower limit (1) (Pollak [3], Tsai et al. [4]). Nonetheless, the most popular control chart is still Shewhart. In spite of its lesser efficiency, its simplicity makes it easy to apply in practice since unlike CUSUM and Shiryaev-Roberts control charts, it does not require sophisticated computer programs. This research focuses on recently proposed control schemes called Nested Plans. These schemes include two unknown parameters that essentially impact their efficiency. The research presents a way to find the optimal values of the parameters, and shows that Nested Plans with the correct choice of the parameters are almost as efficient as the asymptotically optimal Shiryaev-Roberts and CUSUM control charts and almost as simple in practical application as Shewhart method.

2. General sequential methods

In many common situations, we assume that we survey sequentially independent observations $X_1, X_2, \ldots$. Initially, the observations follow an in-control distribution $F_1\left(x|\theta_1\right)$ with a density function $f_1\left(x|\theta_1\right)$. It is possible that at ν, an unknown point in time, an accident is in effect, causing the distribution of the observations to change to an out-of-control distribution $F_2\left(x|\theta_2\right)$ with a density function $f_2\left(x|\theta_2\right)$, where θ_1 and θ_2 are parameter vectors. In this section we focus on four mentioned in introduction sequential methods. Each method is defined by its stopping time as presented below.

2.1 The Shewhart sequential procedure

The general Shewhart stopping time is

$$N_S = min\left\{n:\ f_2\left(X_n|\theta_2\right)/\ f_1\left(X_n|\theta_1\right)\geq C\right\}\ , \qquad (2)$$

where $C>0$ is a threshold value tuned to satisfy a desired ARL to false alarm. Since the random variable N_S is distributed according to the geometric distribution, the in-control ARL and the out-of-control ARL for the Shewhart method can be straightforwardly calculated as

$$E_{F_1(x|\theta_1)}\left(N_S\right)=\frac{1}{P_{F_1(X|\theta_1)}\left(f_2\left(X|\theta_2\right)/\ f_1\left(X|\theta_1\right)\geq C\right)}\ , \qquad (3)$$

and

$$E_{F_2(x|\theta_1)}\left(N_S\right)=\frac{1}{P_{F_2(X|\theta_1)}\left(f_2\left(X|\theta_2\right)/\ f_1\left(X|\theta_1\right)\geq C\right)}\ , \qquad (4)$$

respectively, where we define by $P_F\left(B\right)$ and $E_F\left(N_S\right)$ the probability of the event B and the expectation of the stopping time N_S under the assumption that the observations come from distribution F .

2.2 Nested Plans as sequential procedures

These schemes consist of two steps: a variable plan and an attributes plan (see Lumelskii et al. [5], Feigin et al. [6]). On the first step, the observations are divided into groups of n observations (including the case $n=1$): $\left(X_{11}, X_{12},\ldots X_{1n}\right)$, $\left(X_{21}, X_{22},\ldots X_{2n}\right)$,...and it is assumed that there is no a change inside these groups. Afterthat, for the group i $(i=1,2,\ldots)$ the Bernoulli variable Z_i is defined as

$$Z_i = \begin{cases} 1 & \text{if } \prod_{j=1}^{n} \frac{f_2\left(X_{ij}|\theta_2\right)}{f_1\left(X_{ij}|\theta_2\right)} \ge C \\ 0 & \text{otherwise} \end{cases}, \qquad (5)$$

where $C>0$ is some threshold and his choice completely specifies the Bernoulli distribution of the random variales $Z_1, Z_2, \ldots$. On the second step, the zero-one observations $Z_1, Z_2, \ldots$ are considered and the out-of-control alarm is triggered if there are 2 ones among the last d observations Z_i, $i=1,2,\ldots$. Thus, the Nested Plans include two parameters: n and d, whose values should be determined before applying the method. The in-control ARL and the out-of-control ARL for the Nested Plan are defined as

$$E_{F_1(x|\theta_1)}\left(N_{NP}\right) = \frac{n\left(2-P_1^{d-1}\right)}{Q_1\left(1-P_1^{d-1}\right)} \qquad (6)$$

and

$$E_{F_2(x|\theta_1)}\left(N_{NP}\right) = \frac{n\left(2-P_2^{d-1}\right)}{Q_2\left(1-P_2^{d-1}\right)}, \qquad (7)$$

respectively, where N_{NP} is a stopping time, $P_h = P\left(Z_i = 0 \middle| X_{ij} \sim F_h\left(x|\theta_h\right), j=1,2,...,n\right)$, $Q_h = 1-P_h$, $h=1,2$.

2.3 The Shiryaev-Roberts and the CUSUM procedures

The Shiryaev-Roberts and the CUSUM stopping times are defined as $N_{SR} = \min\{n: R_n \ge C\}$, $N_{CU} = \min\{n: \Lambda_n \ge C\}$, respectively, where $R_n = \sum_{i=1}^{n} \Delta_n^k$, $\Lambda_n = \max_{1=1,...,n} \Delta_n^k$, $\Delta_n^k = \prod_{i=k}^{n} \frac{f_2\left(X_i|\theta_2\right)}{f_1\left(X_i|\theta_1\right)}$, $C>0$ is a threshold value tuned to satisfy a desired ARL to false alarm. There are no exact analytical results for the in-control ARL and the out-of-control ARL for the Shiryaev-Roberts and the CUSUM procedures. However, the CUSUM procedure has a non-asymptotic optimal property (Moustakides [7]). That is if the pre-change and the post-change distribution of the observations are known, then the CUSUM procedure most rapidly detect a change in distribution among all procedures with a common bound specifying an acceptable rate of false alarms, i.e. in-control ARL. For the Shiryaev-Roberts procedure, an asymptotic (as the in-control ARL $A \to \infty$) optimality has been shown (Pollak [8]). Note that the optimal properties of the Shiryaev-Roberts and the CUSUM procedures hinge on the true pre-change and post-change densities. Since $f_2\left(X|\theta_2\right)$ is usually a representative of possible post-change densities while in practice the true post-change density is unknown, even a small misspecification of $f_2\left(X|\theta_2\right)$ can result in the true out-of-control ARL being very different from its asymptotically optimal value. Moreover, there are many situations where $f_1\left(X|\theta_1\right)$ is unknown, then even a small misspecification of $f_1\left(X|\theta_1\right)$ can result in the true in-control ARL being very different from the nominal one (Pollak [3]).

3. *A Comparison of the out-of-control ARLs of considered methods*

In this section we compare the out-of-control ARL of the Nested Plans with that of the Shewhart method and asymptotically optimal methods, where the in-control ARL is the same for all methods. For simplicity we assume that the pre-change and the post-change distributions are $F_1\left(x|\theta_1\right) = N\left(\mu_1, \sigma^2\right)$ and $F_2\left(x|\theta_2\right) = N\left(\mu_2, \sigma^2\right)$, respectively, where $\mu_2 = \mu_1 + r\sigma$, $r>0$. Then, by equation (2), the Shewhart stopping time is $N_S = \min\{n: X_n \ge C\}$ and its in-control and out-of-control ARLs are defined according to equations (3), (4), as $E_{N\left(\mu_1,\sigma^2\right)}\left(N_S\right) = 1/\left(1-\Phi\left(\left(C-\mu_1\right)/\sigma\right)\right)$ and $E_{N\left(\mu_2,\sigma^2\right)}\left(N_S\right) = 1/\left(1-\Phi\left(\left(C-\mu_2\right)/\sigma\right)\right)$ respectively, where $\Phi(x)$ is the standard normal cumulative distribution function. In particular, if $E_{N\left(\mu_1,\sigma^2\right)}\left(N_S\right) = A$, then $C = \sigma Z_{1-1/A} + \mu_1$, where $Z_{1-\alpha} = \Phi^{-1}\left(1-\alpha\right)$, $0<\alpha<1$, and

$$E_{N\left(\mu_2,\sigma^2\right)}\left(N_S\right) = \frac{1}{1-\Phi\left(Z_{1-1/A} - r\right)}. \qquad (8)$$

The stopping time of the Nested Plan is defined as in section 2.2, where by equation (5), $Z_i = \begin{cases} 1 & \text{if } \bar{X}_i \ge C \\ 0 & \text{otherwise} \end{cases}$, $\bar{X}_i = \frac{\sum_{j=1}^{n} X_{ij}}{n}$. Then, the in-control and the out-of-control ARLs of the Nested Plan are given by equations (6) and (7), respectively, where $P_h = P_{N\left(\mu_h,\sigma^2\right)}\left(\bar{X}_i < C\right)$, $Q_h = 1-P_h$, $h=1,2$. Thus, straightforwardly, if P_1 is known, then $C = \mu_1 + Z_{P_1}\frac{\sigma}{\sqrt{n}}$,

$$P_2 = \Phi\left(Z_{P_1} - r\sqrt{n}\right). \qquad (9)$$

First, we present a way to determine the parameters n and d of the Nested Plan for a chosen value $r=1$ as the representative of a possible change in the expectation of the assumed normal distribution. Then, determining $r=1$, we compare the out-of-control ARL of the Nested Plan (with a specified parameters) with that of the Shewhart method and asymptotically optimal methods for different values of the in-control ARL. Finally, we consider misspecifications of r and misspecifications of distribution after a possible change and examine the robustness of the Nested Plan and the Shewhart methods.

3.1 Determining parameters of the Nested Plans

Let $r = \left(\mu_2 - \mu_1\right)/\sigma = 1$ be the representative of possible change in the expectation of the assumed normal distribution. Note that, by equation (6), the in-control ARL of the Nested Plan with the specified parameters n and d is the function of one variable P_1, $g\left(P_1\right) = n\left(2-P_1^{d-1}\right)/\left(\left(1-P_1\right)\left(1-P_1^{d-1}\right)\right)$, where $P_1 = P_{N\left(\mu_1,\sigma^2\right)}\left(\bar{X}_i < C\right)$. Since, for any fixed $A>2n$, $g(0) = 2n < A$, $g\left(P_1\right) \xrightarrow[P_1 \to 1]{} \infty$ and $g\left(P_1\right)$ is the increasing function of P_1, the equation $g\left(P_1\right) = A$ has a unique solution that can be obtained numerically. Then, using this solution, the values of $P_2 = P_{N\left(\mu_2,\sigma^2\right)}\left(\bar{X}_i < C\right)$ and the out-of-control ARL of the Nested Plan can be calculated by equation (9) and (7), respectively. The following Tables 1 and 2 present values of the out-of-control ARL of the Nested Plan with the in-control ARL $A=500$ and 1000, respectively, for different parameters: $n \in [1,...,10]$, $d \in [1,...,8]$.

Table 1: Out-of-control ARLs of the Nested Plan, where $A = 500$, $r = 1$.

n/d	2	3	4	5	6	7	8
1	20.62	18.81	18.28	18.14	18.16	18.26	18.40
2	13.75	12.89	12.84	12.99	13.22	13.47	13.73
3	12.34	**11.83**	11.96	12.22	12.50	12.78	13.05
4	12.39	12.08	12.29	12.59	12.87	13.13	13.37
5	13.10	12.92	13.17	13.45	13.70	13.92	14.12
6	14.20	14.10	14.59	14.59	14.81	14.99	15.14
7	15.55	15.50	15.94	15.94	16.11	16.25	16.38
8	17.08	17.07	17.43	17.43	17.57	17.68	17.78
9	18.74	18.75	19.04	19.04	19.15	19.24	19.31
10	20.50	20.53	20.75	20.75	20.83	20.90	20.96

Note that for $r = 1$, by equation (8), the out-of-control ARL of the Shewhart procedure with the in-control ARL $A = 500$ is equal to $E_{N(\mu_2,\sigma^2)}(N_S) = 33.27$. Moreover, for the considered case, the Kullback–Leibler information quantity is $E_{N(\mu_2,\sigma^2)}\left(\log\left(f_{N(\mu_2,\sigma^2)}(X)/f_{N(\mu_1,\sigma^2)}(X)\right)\right) = r^2/2 = 0.5$. Therefore, by equation (1), the asymptotic minimum out-of-control ARL over all possible procedures is approximately equal to $\log 500/0.5 = 12.43$.

Table 2: Out-of-control ARLs of the Nested Plan, where $A = 1000$, $r = 1$.

n/d	2	3	4	5	6	7	8
1	30.69	27.55	26.48	26.05	26.90	25.89	25.96
2	18.10	16.66	16.41	16.49	16.69	16.94	17.21
3	15.02	14.17	14.21	14.45	14.75	15.07	15.38
4	14.29	**13.74**	13.92	14.23	14.55	14.87	15.16
5	14.53	14.17	14.41	14.74	15.05	15.33	15.58
6	15.29	15.06	15.33	15.64	15.91	16.15	16.36
7	16.39	16.25	16.52	16.79	17.02	17.22	17.39
8	17.73	17.65	17.90	18.12	18.31	18.47	18.61
9	19.24	19.21	19.42	19.60	19.75	19.88	19.99
10	20.88	20.87	21.05	21.20	21.32	21.42	21.51

For this case, the out-of-control ARL of the Shewhart procedure is equal to $E_{N(\mu_2,\sigma^2)}(N_S) = 54.62$, and the asymptotic minimum out-of-control ARL over all possible procedures is approximately equal to $\log 1000/0.5 = 13.82$. Tables 1 and 2 show that the out-of control ARL of the Nested Plan is strongly depends on the parameters n and d. However, even for the worst choice of the parameters this procedure is more efficient than the Shewhart's method and for the best choice of the parameters the Nested Plan is comparable with the CUSUM and the Shiryaev-Roberts procedures.

3.2 Analysis of the robustness of the Shewhart and the Nested Plan Methods

Table 3 below presents out-of-control ARLs of the Shewhart and the Nested Plan methods with the in-control ARL=500,1000, for values of $r = (\mu_2 - \mu_1)/\sigma$ that are different from its representative value $r = 1$. Note that the parameters of the Nested Plan were obtained based on this representative value.

Table 3: Out-of-control ARLs of the Shewhart and the Nested Plan procedures

In-control ARL=500		
r	**Nested Plan (n=3, d=3)**	**SHEWHART**
0.7	22.77	68.49
0.8	17.99	53.19
0.9	14.85	41.84
1	11.83	33.27
1.1	11.26	26.67
1.2	10.24	21.50
1.3	9.53	17.51
In-control ARL=1000		
r	**Nested Plan (n=4, d=3)**	**SHEWHART**
0.7	26.30	119.05
0.8	20.26	90.91
0.9	16.35	69.93
1	13.74	54.62
1.1	11.96	42.92
1.2	10.73	34.01
1.3	9.87	27.24

Table 3 shows that the Nested Plan is much more effective than the Shewhart procedure for all values of r and the gap between out-of-control ARLs of the Nested Plan and the Shewhart procedure increase when decreasing the change in expectations of the pre-change and the post-change distributions. The next Table 4 presents out-of-control ARLs of the Shewhart and the Nested Plan methods with the in-control ARL=500,1000, for different post-change distributions. Note that these procedures were specified taking into account that before a possible change the observations come from the standard normal distribution and the post-change distribution is the normal distribution with the expectation $\mu = 1$ and the variance $\sigma^2 = 1$. That is, the Shewhart stopping time was defined as $N_S = \min\{n: X_n \ge Z_{1-1/A}\} = \begin{cases} \min\{n: X_n \ge 2.88\} & \text{if } A = 500 \\ \min\{n: X_n \ge 3.09\} & \text{if } A = 1000 \end{cases}$. For the Nested Plan, by equation (6), the value of P_1 that corresponds to optimal values of the patrameters n and d is $P_1 = \begin{cases} 0.9413 & \text{if } A = 500 \quad (n=3, d=3) \\ 0.9527 & \text{if } A = 1000 \quad (n=4, d=3) \end{cases}$. The procedure raises an alarm if there are 2 ones among the last d observations Z_i, $i = 1, 2, \ldots$, where for $A = 500$, $Z_i = \begin{cases} 1 & \bar{X}_i \ge 1.5658/\sqrt{3} \\ 0 & \text{oherwise} \end{cases}$, $\bar{X}_i = \frac{1}{3}\sum_{j=1}^{3} X_{ij}$ and for $A = 1000$, $Z_i = \begin{cases} 1 & \bar{X}_i \ge 1.6716/2 \\ 0 & \text{oherwise} \end{cases}$, $\bar{X}_i = \frac{1}{4}\sum_{j=1}^{4} X_{ij}$. Thus, for a real post-change distribution F, the out-of-control ARL of the Shewhart method is defined as $E_F(N_S) = \begin{cases} 1/P_F(X \ge 2.88) & \text{if } A{=}500 \\ 1/P_F(X \ge 3.09) & \text{if } A{=}1000 \end{cases}$. The out-of-control ARL of the Nested Plan is given by equation (7), where for $A = 500$, $P_2 = P_F\left(\bar{X}_3 < 1.5658/\sqrt{3}\right)$, $d = 3$ and for $A = 1000$, $P_2 = P_F\left(\bar{X}_4 < 1.6716/2\right)$, $d = 3$. Table 4 considers cases where the post-change distribution F is a distribution of the random variable $X + 1$, $X \sim F^*$ and F^* is a Uniform, Normal, Student and Laplace distribution with different parameters but with zero expectation. That is considered post-change distributions have the expectation 1 and different variances.

Table 4: Out-of-control ARLs of the Shewhart and the Nested Plan procedures

In-control ARL=500		
F^*	**Nested Plan (n=3, d=3)**	**SHEWHART**
Unif(-2,2)	12.29	33.33
Unif(-2.5,2.5)	12.65	8.06
$t_{(3)}$	12.27	12.65
$t_{(5)}$	12.02	16.95
Norm(0,1)	11.83	33.27
Laplace(0,1)	12.17	13.10
Laplace(0,0.7)	11.53	29.33
Laplace(0,0.5)	10.81	85.90
Laplace(0,0.4)	10.25	219.89
Laplace(0,0.3)	9.48	1053.43
In-control ARL=1000		
F^*	**Nested Plan (n=4, d=3)**	**SHEWHART**
Unif(-2.5,2.5)	15.15	12.20
Unif(-3,3)	15.67	6.59
$t_{(3)}$	14.73	15.61
$t_{(5)}$	14.26	22.22
Norm(0,1)	13.74	54.62
Laplace(0,1)	14.54	16.17
Laplace(0,0.7)	13.36	39.60
Laplace(0,0.5)	12.09	130.73
Laplace(0,0.4)	11.26	371.72
Laplace(0,0.3)	10.24	2121.36

Table 4 shows that the Shewhart procedure sometimes is a good tool but may break down completely and almost always inferior to the Nested Plan. Moreover, the Nested plan looks as a stable and robust procedure for all considered situations.

4. *Conclusions*

This research examined different schemes for sequential quality control. In particular, the efficient CUSUM and Shiryaev-Roberts procedures have some optimal properties for changepoint detection. Nonetheless, the most popular control chart is still Shewhart. In spite of its lesser efficiency, its simplicity makes it easy to apply in practice. The paper focuses on recently proposed sequential schemes called Nested Plans. It turned out that this approach provide efficient and robust results. Due to their simplicity for practical applications these methods can be a good alternative to the Shewhart procedure. Hopefully this paper will stimulate future theoretical and applied research on this topic.

5. *References*

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