ANALYTICAL AND NUMERICAL ASPECTS OF THE SOLUTION OF THE PROBLEM OF A VISCOUS WEAKLY COMPRESSIBLE LIQUID MIXTURE MOTION THROUGH THE VERTICAL PIPE OF THE CIRCULAR CROSS-SECTION

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Abstract: The paper considers a way of the numerical solution of a system of partial differential equations describing the nonstationary flow of a viscous liquid along a vertical straight pipe of circular cross-section. The result obtained is not final, because the proposed approximation scheme is the simplest and provides only the first order of accuracy. Computer modeling has shown that such an approximation is suitable only for a small time interval.

Keywords: NON-STATIONARY HYDRODYNAMICS, LIQUID MIXTURES, WEAK COMPRESSIBILITY, VERTICAL PIPE, MATH MODELING, NUMERICAL SOLUTION

1. Introduction

The problem of investigating the motion of a viscous liquid along a vertical pipe arises in the field of extraction of petroleum products. It is necessary to predict the pressure, density and velocity of the mixture that rises from the depth along the pipe. To understand whether, for example, pressure changes are so critical that they will lead to partial destruction of equipment.

It is known that an analytical solution of equations describing real physical processes is possible only for a narrow class of problems (for example, the heat equation or wave equation on a straight line or in simple form regions). In other cases, it is necessary to apply numerical methods to obtain an approximate solution of the problem. Here arise questions of convergence and stability of the numerical algorithm.

2. Mathematical model

Equations, describing the viscous weakly compressible liquid motion along the vertical pipe of the circular cross-section have the following form [1]:

$$\frac{\partial \rho}{\partial t} + \upsilon \frac{\partial \rho}{\partial z} + \rho \frac{\partial \upsilon}{\partial z} = 0 , \qquad (1)$$

$$\frac{\partial^2 \upsilon}{\partial r \partial z} = \frac{\partial p}{\partial r}, \qquad (2)$$

$$\frac{\partial \upsilon}{\partial t} + \upsilon \frac{\partial \upsilon}{\partial z} = -\rho g + \frac{1}{\rho}\left(\mu \left[\frac{\partial^2 \upsilon}{\partial r^2} + \frac{1}{r}\frac{\partial \upsilon}{\partial r} \right] + \lambda \frac{\partial^2 \upsilon}{\partial z^2} - \frac{\partial p}{\partial z} \right). \qquad (3)$$

The system of equations (1)-(3) was obtained on the basis of the basic equations of hydrodynamics [2]. Three equations contain three unknown functions: density $\rho(z)$, pressure $p(z)$ and velocity $\upsilon_z(r,z,t)$. For the unique solvability of these equations it is necessary to have initial and boundary conditions:

- the pressure at the pipe inlet and near the pipe wall:

$$p(r,0,t) = p_0 , \qquad (4)$$

$$\left. \left(\frac{\partial p}{\partial r} + \frac{\partial p}{\partial z} \right) \right|_{r=R} = 0 \qquad (5)$$

- the density at the initial moment of time and at the pipe inlet:

$$\rho(r,z,0) = \rho_0 , \qquad (6)$$

$$\rho(r,0,t) = \rho_0 ; \qquad (7)$$

- initial velocity:

$$\upsilon(r,z,0) = \varphi(r,z); \qquad (8)$$

- velocity at the wall of the pipe:

$$\upsilon(R,z,t) = 0 ; \qquad (9)$$

- velocity at the pipe outlet:

$$\upsilon(r,L,t) = \psi(r,t) . \qquad (10)$$

Suppose that the density is weakly changing with the z coordinate, consequently, in equation (1) the product $\upsilon \frac{\partial \rho}{\partial z}$ may be neglected, since it is very small in comparison with the other terms.

To find the numerical solution of system (1)-(3) we introduce a uniform grid:

$\bar{\omega}_r = \left\{ r_i = i \cdot h_r,\ i = \overline{0,M},\ h_r = \frac{R}{M},\ r_0 = 0,\ r_M = R \right\}$ – grid by variable r,

$\bar{\omega}_z = \left\{ z_j = j \cdot h_z,\ j = \overline{0,N},\ h_z = \frac{L}{N},\ z_0 = 0,\ z_N = L \right\}$ – grid by variable z,

$\bar{\omega}_t = \left\{ t_k = k \cdot h_t,\ k = \overline{0,K},\ h_t = \frac{T}{K},\ t_0 = 0,\ t_K = T \right\}$ – grid by variable t.

To approximate the equations we use finite differences. Consider equation (1) and taking into account the assumption made, we obtain the following difference scheme:

$$\frac{\rho_{i,j}^{k+1} - \rho_{i,j}^{k}}{h_t} + \rho_{i,j}^{k} \frac{\upsilon_{i,j}^{k} - \upsilon_{i,j-1}^{k}}{h_z} = 0 .$$

It is not difficult to obtain

$$\rho_{i,j}^{k+1} = \rho_{i,j}^{k} \left(1 - \frac{\left(\upsilon_{i,j}^{k} - \upsilon_{i,j-1}^{k} \right) h_t}{h_z} \right). \qquad (1')$$

Boundary and initial conditions for the density are approximated exactly:

$$\rho_{i,j}^{0} = \rho_0 , \qquad (6')$$

$$\rho_{i,1}^{k+1} = \rho_0 . \qquad (7')$$

We do the same with the equation (2):

$$p_{i-1,j}^{k+1} = p_{i,j}^{k+1} - \frac{\upsilon_{i+1,j+1}^{k+1} - \upsilon_{i-1,j+1}^{k+1} - \upsilon_{i+1,j-1}^{k+1} + \upsilon_{i-1,j-1}^{k+1}}{4dz}, \qquad (2)$$

and its boundary conditions

$$p_{i,0}^{k} = p_0 , \qquad (4')$$

$$p_{M,j}^{k+1} = \frac{\left(p_{M,j-1}^{k+1} - p_{M-1,j-1}^{k+1} \right) dz}{dr} + p_{M,j-1}^{k+1} . \qquad (5')$$

For the equation (3) we apply the simplest explicit difference scheme:

$$\frac{\upsilon_{i,j}^{k+1} - \upsilon_{i,j}^{k}}{dt} + \upsilon_{i,j}^{k} \frac{\upsilon_{i,j}^{k} - \upsilon_{i,j-1}^{k}}{dz} =$$
$$= -\rho_{i,j}^{k} g + \frac{1}{\rho_{i,j}^{k}} \left(\mu \left[\frac{\upsilon_{i+1,j}^{k} - 2\upsilon_{i,j}^{k} + \upsilon_{i-1,j}^{k}}{dr^2} + \frac{1}{r_i} \frac{\upsilon_{i,j}^{k} - \upsilon_{i-1,j}^{k}}{dr} \right] + \lambda \frac{\upsilon_{i,j+1}^{k} - 2\upsilon_{i,j}^{k} + \upsilon_{i,j-1}^{k}}{dz^2} - \frac{p_{i,j}^{k+1} - p_{i,j-1}^{k+1}}{dz} \right).$$

We express $\upsilon_{i,j}^{k+1}$:

$$\upsilon_{i,j}^{k+1} = -\rho_{i,j}^{k} g dt +$$
$$+\frac{dt}{\rho_{i,j}^{k}}\left(\mu\left[\frac{\upsilon_{i+1,j}^{k}-2\upsilon_{i,j}^{k}+\upsilon_{i-1,j}^{k}}{dr^2}+\frac{1}{r_i}\frac{\upsilon_{i,j}^{k}-\upsilon_{i-1,j}^{k}}{dr}\right]+\lambda\frac{\upsilon_{i,j+1}^{k}-2\upsilon_{i,j}^{k}+\upsilon_{i,j-1}^{k}}{dz^2}-\frac{p_{i,j}^{k+1}-p_{i,j-1}^{k+1}}{dz}\right)- \quad (3')$$
$$-\upsilon_{i,j}^{k}\frac{\upsilon_{i,j}^{k}-\upsilon_{i,j-1}^{k}}{dz}dt+\upsilon_{i,j}^{k}.$$

3. *Numerical experiment*

Equation (1') is solved on the basis of known values of velocity and density at the previous time layer. Equations (2') and (3') need to be solved jointly on the same time layer, but sequentially along the coordinate, because of the boundary conditions. The pressure on the new layer in height is calculated from the pipe wall to its axis.

We set ourselves by simple initial conditions: the Poiseuille distribution for the velocity, the density at all points, the given differential pressure (fig. 1, 2, 3).

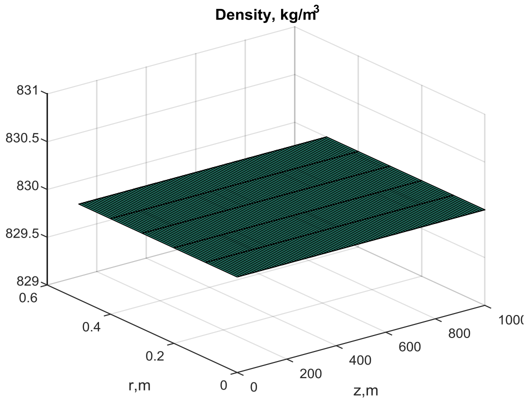

Figure1. The initial density

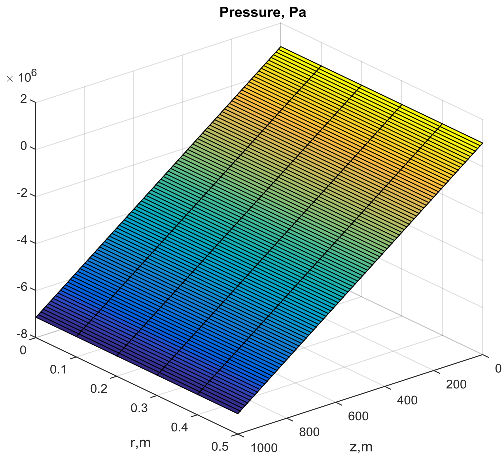

Figure 2. The initial pressure distribution

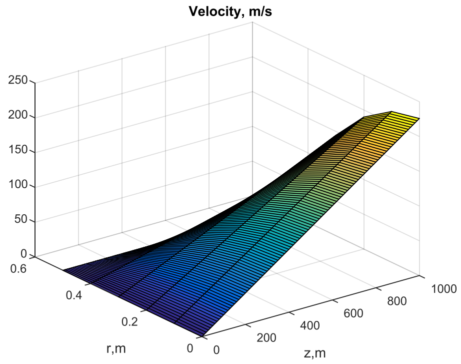

Figure 3. The initial velocity distribution

After the calculations performed, we get the following results (fig.4,5,6)

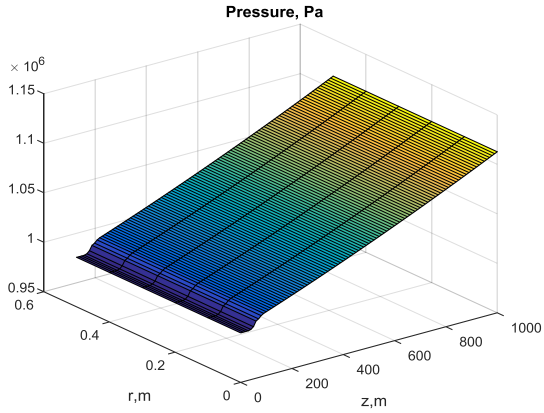

Figure 4. Pressure field

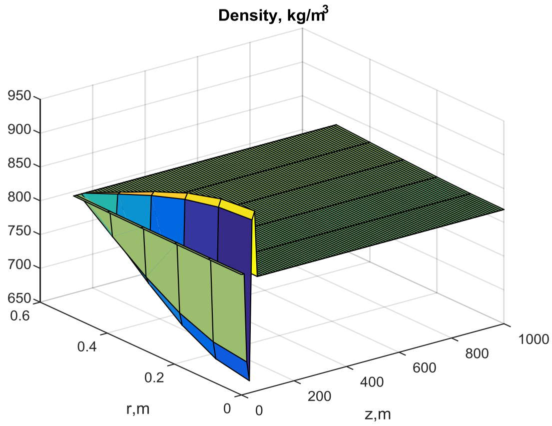

Figure 5. Liquid density

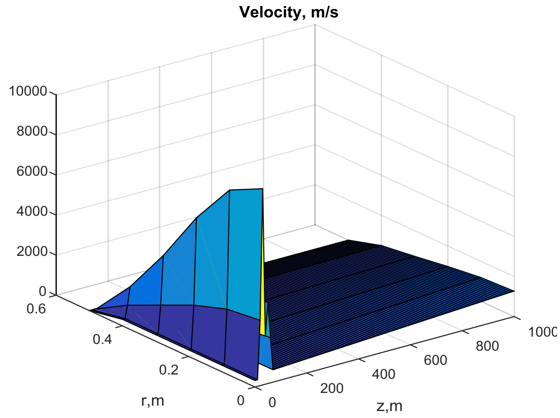

Figure 6. Velocity distribution

The proposed scheme is conditionally stable, and in the problem under consideration, acceptable results are obtained for a small time interval and a small step in time.

In the future, when the equation (2) is approximated, we will most likely have to follow a different path. So this equation can be integrated over r :

$$p(r,z,t)=\frac{\partial \upsilon}{\partial z}+f(z,t) .$$

It is logical to define the function $f(z,t)$ as hydrostatic pressure, which acts in the liquid besides the dynamic pressure в жидкости помимо динамического, and which depends only on the level of liquid lifting, i.e. z coordinate (and implicitly on t).

4. Conclusion

The method described in this paper is only the first approximation to the solution of the system of equations (1)–(3). The main difficulty lies in the fact that differential equations of different orders enter into this system, and equation (3) is also nonlinear. In the future, approximation methods that ensure greater accuracy of these equations will be considered, and solution methods suitable for a system of partial differential equations of different orders.

5. References

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