LAPLACIAN PRESERVING TRANSFORMATION OF SURFACES AND APPLICATION TO BOUNDARY VALUE PROBLEMS FOR LAPLACE'S AND POISSON'S EQUATIONS

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***Abstract**: This paper shows that the constrained similarity transformation of surfaces under boundary constraints is a Laplacian preserving transformation. First, a general proof is presented and then the result is verified for mesh-functions through particular examples. The fact that the constrained similarity transformation, subject to boundary constraints, is a Laplacian preserving transformation is used to construct a method for solving boundary value problems for the Laplace's and the Poisson's equations. Given any solution to these equations we apply the constrained similarity transformation to get the particular solution that satisfies the given boundary conditions.*

Keywords: CONSTRAINED SIMILARITY OF SURFACES, LAPLACIAN PRESERVING TRANSFORMATION, PARTIAL DIFFERENTIAL EQUATIONS, LAPLACE'S EQUATION, POISSON'S EQUATION, BOUNDARY VALUE PROBLEM

1. Introduction

Constrained similarity transformation for functions of one independent variable has been defined in [1] via minimizing the H^1 semi-norm [2] of the functional difference. An exact general solution for mesh-functions has been obtained. The results were used as basis to construct the *shooting-projection* method for solving two-point boundary value problems for second order ordinary differential and integro-differential equations [3], [4]. In [5] the results obtained in [1] have been extended to functions of two independent variables, i.e. to surfaces. An exact general solution for constrained similarity transformation of mesh-surfaces under linear constraints has been obtained. In the current paper these results will be used to construct a method for solving two-point boundary value problems for the Laplace's and the Poisson's equations. In what fallows we briefly summarise the main results for constrained similarity transformation of surfaces.

Let $z=z(x,y)$ and $z^*=z^*(x,y)$ be two real-valued functions of the real independent variables $(x,y)\in G$, where G is bounded domain in R^2. The functions z and z^* define two surfaces. Suppose the surface z is given together with a set of constrains that the surface does not meet. The surface z^* that satisfies the constraints and minimizes

$$\iint_G (\nabla z^* - \nabla z)^2\,dxdy = \iint_G \left(\frac{\partial z^*}{\partial x} - \frac{\partial z}{\partial x}\right)^2 dxdy + \iint_G \left(\frac{\partial z^*}{\partial y} - \frac{\partial z}{\partial y}\right)^2 dxdy, \quad (1)$$

is called *similar* to the surface z under the imposed constraints [5]. In (1) $\nabla=(\partial/\partial x,\ \partial/\partial y)$ is the Nabla operator (gradient). The transformation of z into z^* is called *constrained similarity transformation of surfaces*.

Consider a rectangular domain $G=\{(x,y)|\ x\in[x_a,x_b], y\in[y_a,y_b]\}$. Suppose the intervals $x\in[x_a,x_b]$ and $y\in[y_a,y_b]$ could be partitioned by N_x and N_y mesh-points into N_x-1 and N_y-1 intervals of size h, respectively. The set of points $\{(x_k,y_l),\ x_k=x_a+(k-1)h,\ y_l=y_a+(l-1)h,\ k=1,2,\dots,N_x,\ l=1,2,\dots,N_y\}$ defines a *uniform mesh* on the rectangle. The constant h is the step-size of the mesh. Let the surface $z(x,y)$ be given as *mesh-surface*, i.e. $z=\{z_{k,l}=z(x_k,y_l),\ k=1,2,\dots,N_x,\ l=1,2,\dots,N_y\}$. In order to define constrained similarity between the given surface z and $z^*=\{z^*_{k,l},\ k=1,2,\dots,N_x,\ l=1,2,\dots,N_y\}$ expression (1) is discretized yielding the following *objective function*:

$$I = \sum_{k=1}^{N_x-1}\sum_{l=1}^{N_y}((z^*_{k+1,l} - z^*_{k,l}) - (z_{k+1,l} - z_{k,l}))^2 + \sum_{k=1}^{N_x}\sum_{l=1}^{N_y-1}((z^*_{k,l+1} - z^*_{k,l}) - (z_{k,l+1} - z_{k,l}))^2. \quad (2)$$

To use the formulas derived in [1] the following notation is used: $z_{k,l}=u_{k+(l-1).Nx}$, $z^*_{k,l}=u^*_{k+(l-1).Nx}$ for $k=1,2,\dots,N_x$, $l=1,2,\dots,N_y$ and the vectors $u=[z_{1,1},z_{2,1},..,z_{Nx,1},z_{1,2},z_{2,2},..,z_{Nx,2},..,z_{1,Ny},z_{2,Ny},..,z_{Nx,Ny}]^T$, $u^*=[z^*_{1,1},z^*_{2,1},..,z^*_{Nx,1},z^*_{1,2},z^*_{2,2},..,z^*_{Nx,2},..,z^*_{1,Ny},z^*_{2,Ny},..,z^*_{Nx,Ny}]^T$ are introduced. The minimum of I is sought subject to M linear constraints:

$$\sum_{i=1}^{N} A_{ji} u^*_i = c_j, \quad j = 1,2,\dots,M < N. \quad (3)$$

where $N=N_xN_y$. The constraints (3) are written in a matrix form as $Au^*=c$, where

$$A = \begin{bmatrix} A_{11} & A_{12} & . & . & . & A_{1N} \\ A_{21} & A_{22} & . & . & . & A_{2N} \\ . & . & . & . & . & . \\ A_{M1} & A_{M2} & . & . & . & A_{MN} \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ . \\ c_M \end{bmatrix} \quad (4)$$

and u^* is the $N\times 1$ column-vector of the unknowns. To find u^* that minimizes the objective function (2) and satisfies the constraints (3) the Lagrange's method of the undetermined coefficients [6] is employed obtaining (for derivations see [1] and [5]):

$$u^* = u - (\overline{L} + \overline{A})^{-1}\left(\frac{1}{2}A^T\lambda + \overline{A}u - \overline{c}\right), \quad (5)$$

$$\lambda = 2\left(A(\overline{L} + \overline{A})^{-1}A^T\right)^{-1}\left(Au - c - A(\overline{L} + \overline{A})^{-1}(\overline{A}u - \overline{c})\right), \quad (6)$$

where $\overline{A}$ is the matrix A augmented with $N-M$ rows of zeros to an $N\times N$ matrix, the column-vector $\overline{c}$ is the column-vector c augmented with $N-M$ zeros to an $N\times 1$ column-vector, λ is the $M\times 1$ column-vector of the undetermined coefficients λ_j, $j=1,2,\dots,M$, and $\overline{L}$ is the $N\times N$ $(N=N_xN_y)$ matrix, given as block matrix by:

$$\overline{L} = \overbrace{\begin{bmatrix} L_1 & E & 0 & 0 & 0 & . & . & 0 \\ E & L_2 & E & 0 & 0 & . & . & 0 \\ 0 & E & L_2 & E & 0 & . & . & 0 \\ & & & . & . & & & \\ & & & & . & . & & \\ & & & & & . & . & \\ 0 & . & . & . & 0 & E & L_2 & E \\ 0 & . & . & . & 0 & 0 & E & L_1 \end{bmatrix}}^{N_y}, \quad (7)$$

where

$$L_1 = \overbrace{\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & . & . & 0 \\ 1 & -3 & 1 & 0 & 0 & . & . & 0 \\ 0 & 1 & -3 & 1 & 0 & . & . & 0 \\ & & & . & . & & & \\ & & & & . & . & & \\ & & & & & . & . & \\ 0 & . & . & . & 0 & 1 & -3 & 1 \\ 0 & . & . & . & 0 & 0 & 1 & -2 \end{bmatrix}}^{N_x},$$

$$L_2 = \begin{bmatrix} -3 & 1 & 0 & 0 & 0 & . & . & 0 \\ 1 & -4 & 1 & 0 & 0 & . & . & 0 \\ 0 & 1 & -4 & 1 & 0 & . & . & 0 \\ & & & . & . & & & \\ & & & & . & . & & \\ & & & & & . & . & \\ 0 & . & . & . & 0 & 1 & -4 & 1 \\ 0 & . & . & . & 0 & 0 & 1 & -3 \end{bmatrix},$$

the matrix E is the $N_x \times N_x$ unit matrix, and 0 is the $N_x \times N_x$ zero matrix.

The right-hand side of (6) contains only known quantities. Once the column-vector λ is calculated it is substituted into (5) and u^*_i, $i=1,2,\dots,N$ are obtained. Then u^*_i are converted to $z^*_{k,l}$ and the sought function (surface) is found.

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2. Laplacian preserving transformation of surfaces as constrained similarity under boundary constraints

In this paragraph we prove that the constrained similarity transformation of the surface $z(x,y)$, $(x,y)\in G$ into the surface $z^*(x,y)$, $(x,y)\in G$ under the boundary constraints $z^*(x,y)=z_0(x,y)$, $(x,y)\in \partial G$, where ∂G is the boundary of the domain G and $z_0(x,y)$ is a given continuous function on the boundary, is a *Laplacian preserving transformation*, i.e.

$$\Delta z^* = \Delta z, \;\; (x,y) \in G, \tag{8}$$

where Δ is the Laplace operator (Laplacian):

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{9}$$

Proof: If a function z^* with fixed values at the boundary ∂G should minimize the functional (1) then the following Euler-Lagrange equation for the integrand $(\nabla z^* - \nabla z)^2$ must hold [6], [7]:

$$\frac{\partial}{\partial x}\left(\frac{\partial (\nabla z^* - \nabla z)^2}{\partial (\partial z^*/\partial x)} \right) + \frac{\partial}{\partial y}\left(\frac{\partial (\nabla z^* - \nabla z)^2}{\partial (\partial z^*/\partial y)} \right) - \frac{\partial (\nabla z^* - \nabla z)^2}{\partial z^*} = 0, \tag{10}$$

Taking into account that

$$(\nabla z^* - \nabla z)^2 = \left(\frac{\partial z^*}{\partial x} - \frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z^*}{\partial y} - \frac{\partial z}{\partial y} \right)^2 \tag{11}$$

and performing the differentiation in (10) with respect to $\partial z^*/\partial x$, $\partial z^*/\partial y$, and z^* yields:

$$2\frac{\partial}{\partial x}\left(\frac{\partial z^*}{\partial x} - \frac{\partial z}{\partial x} \right) + 2\frac{\partial}{\partial y}\left(\frac{\partial z^*}{\partial y} - \frac{\partial z}{\partial y} \right) = 0, \; (x,y) \in G, \tag{12}$$

which, after rearranging, is just equation (8).

3. Solving boundary value problems for Poisson's and Laplace's equations using constrained similarity transformation

Let a function $z(x,y)$, $(x,y)\in G$ satisfy the Poisson's equation:

$$\Delta z = -f(x,y), \;\; (x,y) \in G. \tag{13}$$

In the case when $f(x,y)\equiv 0$ the Poisson's equation is just the Laplace's equation so all the conclusions in this paragraph will hold for both equations. The problem of finding a solution $z^*(x,y)$, $(x,y)\in G$ that satisfies (13) and at the same time satisfies the boundary condition

$$z^*(x,y) = z_0(x,y), \;\; (x,y) \in \partial G, \tag{14}$$

where ∂G is the boundary of the domain G and $z_0(x,y)$ is a given continuous function on the boundary, constitutes a *boundary value problem* (BVP), more specifically *Dirichlet problem*, for the Poisson's equation [7], [8]. According to the result proven in the previous paragraph, if any function z that satisfies (13) is subjected to constrained similarity transformation under the boundary constraints (14), then the obtained function z^* will satisfy the boundary constraints (14) *and* the differential equation (8), hence also the Poisson's equation (13). Therefore, the obtained function z^* will be just the sought BVP solution.

In the case when G is a rectangular domain in R^2 and z is any solution to (13) given as mesh-function, then the BVP solution z^* can be obtained (as mesh-function) using formulas (5) and (6).

4. Results

In this paragraph two examples are presented. In Example 1 a particular Poisson's equation is considered and three different BVPs for this equation are solved using the constrained similarity transformation of surfaces. It is verified that the constrained similarity transformation is indeed Laplacian preserving transformation. In Example 2 the harmonic function, i.e. twice continuously differentiable function satisfying the Laplace's equation, is found for three particular boundary conditions. Again, the Laplacian is calculated to verify (8).

Example 1

Consider the function (surface) $z(x,y)=xy(y-x)$ defined on the rectangular domain $G=\{(x,y)|\; x\in[-2,2],\; y\in[-2,2]\}$. The function z satisfies the Poisson's equation:

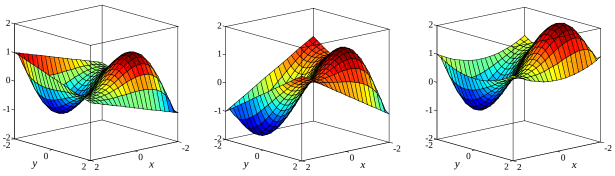
$$\Delta z = -2(y-x), \;\; (x,y) \in G. \tag{15}$$

Find the function z^* that satisfies (15) and at the same time satisfies the boundary condition $z^*(x,y)= z_0(x,y)$, $(x,y)\in\partial G$, where ∂G is the boundary of the domain G. Solve the problem for the following three cases:

$$\text{(a) } z_0(x,y) = \frac{1}{4}(x-y) \quad \text{(b) } z_0(x,y) = \frac{1}{4}xy \quad \text{(c) } z_0(x,y) = \frac{1}{8}(x^2+y^2) \tag{16}$$

To solve the given BVPs first the intervals $x\in[-2,2]$ and $y\in[-2,2]$ are partitioned by $N_x=21$ and $N_y=21$ mesh-points into intervals of size $h=0.2$ and the surface z is calculated on the mesh: $z=\{z_{k,l}=x_k y_l(y_l-x_k)$, $x_k=-2+(k-1)h$, $y_l=-2+(l-1)h$, $k=1,2,\dots,N_x$,

l=1,2,…,N_y}. Then the mesh-surface z is subjected consecutively to the constrained similarity transformation (5)-(6) for the three boundary constraints (16) (a)-(c). The obtained surfaces $z^*=\{z^*_{k,l}$, k=1,2,…,N_x, l=1,2,…,N_y} are shown in Fig. 1. They satisfy the Poisson's equation (15) and the respective boundary conditions (16) (a)-(c), hence they are the sought BVP solutions.



(a) ***(b)*** ***(c)***

Fig.1. *The surface z* satisfying the Poisson's equation (15) for the respective boundary condition (15)* ***(a)****,* ***(b)****, and* ***(c)****.*

To verify that the three surfaces shown in Fig. 1 satisfy the Poisson's equation (15) Δz and Δz^* for case (a), (b), and (c) are calculated for the inner points of the mesh using the central finite difference approximation:

$$\Delta z \approx \frac{z_{k+1,l} + z_{k-1,l} + z_{k,l+1} + z_{k,l-1} - 4z_{k,l}}{h^2}, \qquad (17)$$

where k=2,…,N_x–1, and l=2,…,N_y–1. All four results coincide yielding, within numerical precision, the plane 2(x–y) (see eqn. (15)).

Example 2

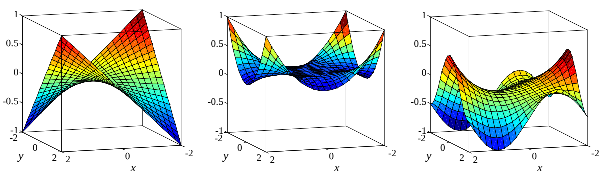
Consider the Laplace's equation:

$$\Delta z = 0,\ \ (x, y) \in G. \qquad (18)$$

Any function $z(x,y)$ which has continuous second derivatives in the domain G and satisfies the Laplace's equation (18) is called *harmonic* in G [7], [8]. The plane $z(x,y)$=0, $(x,y)\in G$ is obviously harmonic. In this example we find the harmonic function $z^*(x,y)$ that satisfies the boundary condition $z^*(x,y)= z_0(x,y)$, $(x,y)\in\partial G$, where ∂G is the boundary of the domain $G=\{(x,y)|\ x\in[-2,2], y\in[-2,2]\}$. The problem is solved for the following three cases:

$$\text{(a)}\ \ z_0(x,y) = \frac{1}{4}xy \qquad \text{(b)}\ \ z_0(x,y) = \frac{1}{4}(x^2+y^2) - 1$$
$$\text{(c)}\ \ z_0(x,y) = \frac{1}{2}\left(\cos\left(\frac{\pi}{2}y\right) - \sin\left(\frac{\pi}{2}x\right)\right) \qquad (19)$$

To solve the problem, as in Example 1, the intervals $x\in[-2,2]$ and $y\in[-2,2]$ are partitioned by N_x=21 and N_y=21 mesh-points into intervals of size h=0.2 and the plane $z(x,y)$=0, $(x,y)\in G$ is defined on the mesh: $z=\{z_{k,l}=0,\ x_k=-2+(k-1)h,\ y_l=-2+(l-1)h,\ k=1,2,\ldots,N_x,$ l=1,2,…,N_y}. Then the mesh-plane z is subjected to the constrained similarity transformation (5)-(6) for the three boundary constraints (19) (a)-(c). The obtained surfaces $z^*=\{z^*_{k,l}$, k=1,2,…,N_x, l=1,2,…,N_y} are shown in Fig. 2. They satisfy the Laplace's equation (18) and the respective boundary conditions (19) (a)-(c).



(a) ***(b)*** ***(c)***

Fig.2. *The three surfaces z* satisfying the Laplace's equation (18) for the respective boundary condition (19)* ***(a)****,* ***(b)****, and* ***(c)****.*

To verify numerically that the three functions z^* shown in Fig. 2 are harmonic, first the partial derivatives for the inner points are calculated using the central finite difference approximation:

$$\frac{\partial^2 z^*}{\partial x^2} \approx \frac{z^*_{k+1,l} - 2z^*_{k,l} + z^*_{k-1,l}}{h^2},\ \ \frac{\partial^2 z^*}{\partial y^2} \approx \frac{z^*_{k,l+1} - 2z^*_{k,l} + z^*_{k,l-1}}{h^2}, \qquad (20)$$

where k=2,…,N_x–1, and l=2,…,N_y–1. Then, using the calculated partial derivatives, the Laplacian $\Delta z^*=\partial^2z^*/\partial x^2+\partial^2z^*/\partial y^2$ for case (a), (b), and (c) is calculated for the inner points. The obtained results indicate that for all three cases the derivatives (20) are continuous and $|\Delta z^*(x,y)|<10^{-11}$ for all $(x,y)\in G$ (i.e. zero within the numerical precision). Therefore, the obtained functions z^* (Fig. 2) are indeed harmonic.

5. Conclusion

It was proven that the constrained similarity transformation of surfaces is a Laplacian preserving transformation. This fact was used to construct a method for solving boundary value problems for Poisson's and Laplace's equations on rectangular domain when any solution to the respective equation is present. Several examples were solved verifying that the Laplacian of the surface is indeed preserved when the surface is subjected to constrained similarity transformation under boundary constraints.

6. Appendix

In this appendix a MATLAB code for solving Example 1(a) is presented. The code could easily be adjusted to other cases. The variables `A_`, `c_`, and `L_` are used for $\overline{A}$, $\overline{c}$, and $\overline{L}$, while `zs`, `us`, `z0`, and `u0` are used for z^*, u^*, z_0, and u_0. The variable `lambda` is used for λ. To define the needed vectors and matrices first the corresponding vectors and matrices composed of zeros and having the required size are defined.

```
function main

    Nx=21; Ny=21; xa=-2; ya=-2; h=0.2;
    M=2*Nx+2*Ny-4; N=Nx*Ny;

    x=zeros(Nx,1); y=zeros(Ny,1);
    for k=1:Nx
        x(k)=xa+h*(k-1);
    end
    for l=1:Ny
        y(l)=ya+h*(l-1);
    end

    z=zeros(Nx,Ny); z0=zeros(Nx,Ny);
    u=zeros(N,1); u0=zeros(N,1);
    i=1;
    for l=1:Ny
        for k=1:Nx
            z(k,l)=x(k)*y(l)*(y(l)-x(k));
            u(i)=z(k,l);
            z0(k,l)=(1/4)*(x(k)-y(l));
            u0(i)=z0(k,l);
            i=i+1;
        end
    end

    A=zeros(M,N); c=zeros(M,1);
    j=1;
    for k=1:Nx
        i=k; A(j,i)=1; c(j)=u0(i); j=j+1;
    end
    for l=2:Ny-1
        i=Nx*(l-1)+1; A(j,i)=1; c(j)=u0(i); j=j+1;
        i=Nx*(l-1)+Nx; A(j,i)=1; c(j)=u0(i); j=j+1;
    end
    for k=1:Nx
        i=Nx*(Ny-1)+k; A(j,i)=1; c(j)=u0(i); j=j+1;
    end
```

```
    A_=zeros(N,N); c_=zeros(N,1);
    for j=1:M
        c_(j)=c(j);
        for i=1:N
            A_(j,i)=A(j,i);
        end
    end

    L_=zeros(N,N);
    L_(1,1)=-2; L_(Nx,Nx)=-2;
    L_(N-Nx+1,N-Nx+1)=-2; L_(N,N)=-2;
    for n=2:(Nx-1)
        L_(n,n)=-3; L_(N-Nx+n,N-Nx+n)=-3;
        L_(n+1,n)=1; L_(N-Nx+n+1,N-Nx+n)=1;
        L_(n-1,n)=1; L_(N-Nx+n-1,N-Nx+n)=1;
        L_(n+Nx,n)=1; L_(N-Nx+n-Nx,N-Nx+n)=1;
    end
    L_(2,1)=1; L_(Nx-1,Nx)=1;
    L_(1+Nx,1)=1; L_(Nx+Nx,Nx)=1;
    L_(N-Nx+2,N-Nx+1)=1; L_(N-Nx+1-Nx,N-Nx+1)=1;
    L_(N-1,N)=1; L_(N-Nx,N)=1;
    for n=Nx+1:N-Nx
        L_(n,n)=-4; L_(n+1,n)=1; L_(n-1,n)=1;
        L_(n+Nx,n)=1; L_(n-Nx,n)=1;
    end
    for n=1:Ny-2
        L_(Nx*n+1,Nx*n+1)=-3;
        L_(Nx*n+Nx,Nx*n+Nx)=-3;
        L_(Nx*n,Nx*n+1)=0;
        L_(Nx*n+Nx+1,Nx*n+Nx)=0;
    end

    H=inv(L_+A_); d=A_*u-c_;

    lambda=(A*H*A')\(A*u-c-A*H*d)*2;
    us=u-H*(A'*lambda/2+d);

    i=1;
    for l=1:Ny
        for k=1:Nx
            zs(k,l)=us(i);
            i=i+1;
        end
    end

    surface(x,y,zs');

end
```

7. References

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