MATHEMATICAL MODELING OF NON-STATIONARY FLOWS OF LIQUID HOMOGENEOUS VISCOUS MIXTURES BY PIPELINES

МАТЕМАТИЧЕКОЕ МОДЕЛИРОВАНИЕ НЕСТАЦИОНАРНЫХ ТЕЧЕНИЙ ЖИДКИХ ОДНОРОДНЫХ СМЕСЕЙ ПО ТРУБОПРОВОДАМ

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Abstract: *In the work a mathematical model for the non-stationary motion of liquid homogeneous viscous mixtures through pipelines is constructed. The corresponding integral equations expressing the laws of conservation of mass, momentum and energy are deriving, from which, in turn, we get the corresponding differential equations. The formulation of the initial and boundary conditions is given; the necessary additional relations that close the corresponding system of differential equations are indicated. A numerical algorithm for solving such a system is proposed. The corresponding numerical examples are given.*

Keywords: PIPELINES, HOMOGENEOUS VISCOUS MIXTURES, NONSTATIONARY MOTION, MATHEMATICAL MODELS, NUMERICAL ANALYSIS

1. Introduction

Trunk pipeline systems have become widespread in various areas of the economy, in particular in the oil and gas and oil refining industries.

Therefore, the task of improving the quality of control of such systems is an important task. One of the most effective ways to solve this problem is to improve the methods of mathematical modeling of the modes of pipeline systems and models for controlling stationary and transient modes of pipeline systems. Approach to the solution of these problems requires the development of mathematical models of pipe networks for numerical analysis of modes and mathematical modeling of control devices, providing the predetermined conditions of viscous liquids in pipe networks.

The difficulty of modeling for the analysis and synthesis of controls for such systems is determined by the complexity and variety of the dynamic description of the flow of viscous liquids in pipelines, as well as the need to take into account many additional factors. On the other hand, the mathematical modeling of pipeline networks is based on the problem of analyzing the solvability of the corresponding initial-boundary value problems for the differential equations of motion of a viscous liquid along a circular pipe, and also on the problem of choosing numerical algorithms for constructing the corresponding solutions. It is natural to base the mathematical modeling of the corresponding processes on the Navier-Stokes equations, but there is no exhaustive (from the mathematical point of view) analysis of the correctness of the initial-boundary value problems for these equations in publications available to authors. Moreover, for real processes, the Navier-Stokes equations are, as a rule, only "first approximation". Therefore, in practice, various simplified mathematical models are used by introducing additional hypotheses about the properties and nature of the motion of liquids in a particular problem. A fairly comprehensive review of simplified mathematical models describing the motion of viscous liquids through pipelines can be found, for example, in [1-4].

In this paper, we do not pretend to be exhaustive, but try to present a solution of the initial-boundary problem of the motion of a viscous liquid along a horizontal cylindrical tube without traditional simplifying hypotheses, relying on the full Navier-Stokes equation, and assuming only the cylindrical symmetry of the problem.

2. Basic equations and statement of the problem

In the basis of the investigation, we shall set the general equations of continuum mechanics in differential form (see, for example, [5]):

$$\frac{d\rho}{dt} + \rho\left(\vec{\nabla}\cdot\vec{\upsilon}\right) = q\,, \tag{1}$$

$$\rho\frac{d\vec{\upsilon}}{dt} = \rho\vec{F} + \frac{\partial\vec{\tau}_x}{\partial x} + \frac{\partial\vec{\tau}_y}{\partial y} + \frac{\partial\vec{\tau}_z}{\partial z}\,, \tag{2}$$

$$\rho\frac{d\vec{M}}{dt} = \rho\vec{\Pi} + \vec{i}\times\vec{\tau}_x + \vec{j}\times\vec{\tau}_y + \vec{k}\times\vec{\tau}_z + \frac{\partial\vec{\pi}_x}{\partial x} + \frac{\partial\vec{\pi}_y}{\partial y} + \frac{\partial\vec{\pi}_z}{\partial z}\,, \tag{3}$$

$$\rho\frac{dE}{dt} = \varepsilon + \vec{\tau}_x\cdot\frac{\partial\vec{\upsilon}}{\partial x} + \vec{\tau}_y\cdot\frac{\partial\vec{\upsilon}}{\partial y} + \vec{\tau}_z\cdot\frac{\partial\vec{\upsilon}}{\partial z} + \vec{\nabla}\cdot\vec{t}\,. \tag{4}$$

Here: $\vec{i}, \vec{j}, \vec{k}$ – are unit coordinate vectors of the corresponding Cartesian coordinate system; $\rho = \rho(x, y, z, t)$ – a liquid density at a point with Cartesian coordinates (x, y, z) at time t; $\vec{\upsilon} = \vec{\upsilon}(x, y, z, t) = \upsilon_x\vec{i} + \upsilon_y\vec{j} + \upsilon_z\vec{k}$ – the liquid velocity at a point (x, y, z) at time t; $\vec{t} = \vec{t}(x, y, z, t) = t_x\vec{i} + t_y\vec{j} + t_z\vec{k}$ – a heat flux vector; $\vec{F} = \vec{F}(x, y, z, t)$ – a (external) mass force per unit mass, acting in the fluid; $q = q(x, y, z, t)$ – a density of mass sources in the liquid per time unit; E – a mass density of the liquid internal energy; ε –a rate of volume absorption (excretion) of energy in the fluid; t_n – a rate of internal energy transmission in a continuous medium through a unit area with a normal $\vec{n}$, i.e. the $dQ_S = dt\oiint_S t_n ds$ is the amount of energy (heat) have passed within time dt through the closed (virtual) surface S (with the external normal $\vec{n}$), bounding some liquid volume; $\vec{M} = \vec{M}(x, y, z, t)$ – an internal angular momentum of the fluid, which a unit of liquid mass possesses; $\vec{\Pi} = \vec{\Pi}(x, y, z, t)$ – an internal angular momentum per mass unit and time unit, which is generated in a volume unit of the liquid; $\vec{\pi}_n = \vec{\pi}_n(x, y, z, t)$ – an internal angular momentum per area unit with the normal $\vec{n}$, penetrating through a (virtual) surface in the liquid per time unit.

$$\vec{T} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$ – stress tensor in the liquid;

$$\vec{\tau}_x = \tau_{xx}\vec{i} + \tau_{xy}\vec{j} + \tau_{xz}\vec{k},$$
$$\vec{\tau}_y = \tau_{yx}\vec{i} + \tau_{yy}\vec{j} + \tau_{yz}\vec{k}, \quad \tau_{\alpha\beta} \equiv \tau_{\alpha\beta}(x,y,z,t);$$
$$\vec{\tau}_z = \tau_{zx}\vec{i} + \tau_{zy}\vec{j} + \tau_{zz}\vec{k},$$
$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \upsilon_x\frac{\partial}{\partial x} + \upsilon_y\frac{\partial}{\partial y} + \upsilon_z\frac{\partial}{\partial z} \equiv \frac{\partial}{\partial t} + \vec{\upsilon}\cdot\vec{\nabla};$$
$$\vec{\nabla} \equiv \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}.$$

Equations (1)-(4) should be taken as a basis for mathematical modeling of a liquid continuous medium. However, this system is not complete.

Functions $\vec{F}, \vec{\Pi}, \varepsilon$, as a rule, are known, or relatively easy to establish empirically; equations (1)-(4) in total are 8. The number of required functions – 28. Hence it follows that in the solution of any problem connected with the motion of a liquid, in each particular case, it is necessary to complement the main system (1)-(4), based on some reasonable assumptions,. This also applies to the correct formulation of the corresponding initial and boundary conditions.

Thus, it makes no sense to talk about the problem of a correct solution of system (1)-(4) in itself (without additional conditions of a non-mechanical nature).

Additional relationships are needed that specify the physicochemical properties of the continuous medium under study. Such relationships are, as a rule, related to the assumptions about the structural and physicochemical features of the continuous medium under investigation, and are based, as indicated above, on empirical considerations and experimental studies.

For example, for an ideal liquid, the stress tensor contains only one unknown function (pressure), instead of nine, for a Newtonian fluid-the stress tensor is symmetric, and its components are expressed in terms of pressure and velocity; the Mendeleev-Clapeyron equation or the adiabatic equation is used for ideal gases. For a liquid that obeys the law of Fourier thermal conductivity, the heat flux vector (three components) is expressed in terms of temperature (one unknown function), and so on.

However, such additional relationships are not related to hydrodynamics as such, but to internal features of the structure of matter and, in addition, as a rule, while they are purely empirical or semi empirical in nature.

In connection with what has been said, we shall formulate the preliminary conditions that we in this paper will base on the exact formulation and solution of the problem.

1) The liquid is assumed to be viscous, isotropic, Newtonian and obeying the Fourier thermal conductivity law. In this case, as is known, the following relations hold:

$$\tau_{xx} = -p + \lambda(\vec{\nabla}\cdot\vec{\upsilon}) + 2\mu\frac{\partial\upsilon_x}{\partial x}, \tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial\upsilon_x}{\partial y} + \frac{\partial\upsilon_y}{\partial x}\right),$$
$$\tau_{yy} = -p + \lambda(\vec{\nabla}\cdot\vec{\upsilon}) + 2\mu\frac{\partial\upsilon_y}{\partial y}, \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial\upsilon_y}{\partial z} + \frac{\partial\upsilon_z}{\partial y}\right), \quad (5)$$
$$\tau_{zz} = -p + \lambda(\vec{\nabla}\cdot\vec{\upsilon}) + 2\mu\frac{\partial\upsilon_z}{\partial z}, \tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial\upsilon_z}{\partial x} + \frac{\partial\upsilon_x}{\partial z}\right),$$

$$t_x = k\frac{\partial T}{\partial x}, \quad t_y = k\frac{\partial T}{\partial y}, \quad t_z = k\frac{\partial T}{\partial z} \quad (6)$$

2) $f(p, \rho, T) = 0$ – is an equation of state of the liquid; in particular, for weakly compressible liquids $\rho = \rho_0\left(1 + \frac{p - p_0}{K}\right)$,

where K is a bulk modulus of the liquid. In particular, experimental data and general physical considerations show, that any medium at a very high temperatures and pressure practically has the ideal liquid properties.

3) Finally, it is assumed, that the following relations are known: $E = E(p,T), \quad \lambda = \lambda(p,T), \quad \mu = \mu(p,T), \quad k = k(p,T)$,

And in the first approximation one can take $\lambda = -\frac{2}{3}\mu$.

In the present paper we consider the case of a *homogeneous incompressible* $(\rho = const)$ *viscous liquid with a constant viscosity coefficient* μ, and $E = c_V T + const$ (here c_V – is the heat capacity for a constant volume). In this case, equations (1), (2) and (4) will be transformed into

$$\frac{\partial\upsilon_x}{\partial x} + \frac{\partial\upsilon_y}{\partial y} + \frac{\partial\upsilon_z}{\partial z} = 0, \quad (7)$$

$$\frac{\partial\upsilon_x}{\partial t} + \upsilon_x\frac{\partial\upsilon_x}{\partial x} + \upsilon_y\frac{\partial\upsilon_x}{\partial y} + \upsilon_z\frac{\partial\upsilon_x}{\partial z} = F_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\Delta\upsilon_x,$$
$$\frac{\partial\upsilon_y}{\partial t} + \upsilon_x\frac{\partial\upsilon_y}{\partial x} + \upsilon_y\frac{\partial\upsilon_y}{\partial y} + \upsilon_z\frac{\partial\upsilon_y}{\partial z} = F_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\Delta\upsilon_y, \quad (8)$$
$$\frac{\partial\upsilon_z}{\partial t} + \upsilon_x\frac{\partial\upsilon_z}{\partial x} + \upsilon_y\frac{\partial\upsilon_z}{\partial y} + \upsilon_z\frac{\partial\upsilon_z}{\partial z} = F_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\Delta\upsilon_z,$$
$$\nu = \frac{\mu}{\rho}, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$c_V\rho\left(\frac{\partial T}{\partial t} + \upsilon_x\frac{\partial T}{\partial x} + \upsilon_y\frac{\partial T}{\partial y} + \upsilon_z\frac{\partial T}{\partial z}\right) = = \varepsilon + \Phi + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right), \quad (9)$$

where

$$\Phi = \mu\left[\begin{array}{c} 2\left(\frac{\partial\upsilon_x}{\partial x}\right)^2 + 2\left(\frac{\partial\upsilon_y}{\partial y}\right)^2 + 2\left(\frac{\partial\upsilon_z}{\partial z}\right)^2 + \\ + \left(\frac{\partial\upsilon_x}{\partial y} + \frac{\partial\upsilon_y}{\partial x}\right)^2 + \left(\frac{\partial\upsilon_y}{\partial z} + \frac{\partial\upsilon_z}{\partial y}\right)^2 + \left(\frac{\partial\upsilon_x}{\partial z} + \frac{\partial\upsilon_z}{\partial x}\right)^2 \end{array}\right].$$

We call attention to the fact, that equations (7) and (8) form a closed system: the number of unknowns ($p, \upsilon_x, \upsilon_y, \upsilon_z$) and the number of equations coincide. Therefore, under our assumptions the system of equations (7) and (8) can be solved independently of the energy equation (9). Among other things we note that equations (8) – are the Navier-Stokes equations for the case under considerations.

Bearing in mind all that has been said above, we narrow down our problem in the following way. To investigate the motion of an isotropic viscous incompressible Newtonian fluid along a horizontal cylindrical tube of constant cross section and finite length, neglecting the action of mass forces.

In solving this problem we will base on equations (7), (8). For definiteness we take the axis of a straight-line horizontal cylindrical pipe of radius R and length l as z-axis. The liquid fills the entire pipe volume; on the left end of the pipe (at the origin of body axes) there is a pump, that sets the liquid pressure, in general, variable in time and equal in cross-section.

In this formulation it is natural to turn to cylindrical coordinate system. In these coordinates the basic equations (7) and (8) will be written as follows [6]:

$$\frac{\partial\upsilon_r}{\partial r} + \frac{1}{r}\frac{\partial\upsilon_\theta}{\partial\theta} + \frac{\partial\upsilon_z}{\partial z} + \frac{\upsilon_r}{r} = 0 \quad (11)$$

$$\begin{aligned}&\frac{\partial \upsilon_r}{\partial t}+\upsilon_r\frac{\partial \upsilon_r}{\partial r}+\frac{\upsilon_\theta}{r}\frac{\partial \upsilon_r}{\partial \theta}+\upsilon_z\frac{\partial \upsilon_r}{\partial z}-\frac{\upsilon_\theta^2}{r}=\\ &=F_r-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\Big(\frac{\partial^2 \upsilon_r}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 \upsilon_r}{\partial \theta^2}+\frac{\partial^2 \upsilon_r}{\partial z^2}+\\ &+\frac{1}{r}\frac{\partial \upsilon_r}{\partial r}-\frac{2}{r^2}\frac{\partial \upsilon_\theta}{\partial \theta}-\frac{\upsilon_r}{r^2}\Big),\end{aligned} \tag{12}$$

$$\begin{aligned}&\frac{\partial \upsilon_\theta}{\partial t}+\upsilon_r\frac{\partial \upsilon_r}{\partial r}+\frac{\upsilon_\theta}{r}\frac{\partial \upsilon_\theta}{\partial \theta}+\upsilon_z\frac{\partial \upsilon_\theta}{\partial z}+\frac{\upsilon_r\upsilon_\theta}{r}=\\ &=F_\theta-\frac{1}{\rho r}\frac{\partial p}{\partial \theta}+\nu\begin{pmatrix}\dfrac{\partial^2 \upsilon_\theta}{\partial r^2}+\dfrac{1}{r^2}\dfrac{\partial^2 \upsilon_\theta}{\partial \theta^2}+\dfrac{\partial^2 \upsilon_\theta}{\partial z^2}+\\ +\dfrac{1}{r}\dfrac{\partial \upsilon_\theta}{\partial r}-\dfrac{2}{r^2}\dfrac{\partial \upsilon_r}{\partial \theta}-\dfrac{\upsilon_\theta}{r^2}\end{pmatrix},\end{aligned} \tag{13}$$

$$\begin{aligned}&\frac{\partial \upsilon_z}{\partial t}+\upsilon_r\frac{\partial \upsilon_z}{\partial r}+\frac{\upsilon_\theta}{r}\frac{\partial \upsilon_z}{\partial \theta}+\upsilon_z\frac{\partial \upsilon_z}{\partial z}=\\ &=F_z-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\begin{pmatrix}\dfrac{\partial^2 \upsilon_z}{\partial r^2}+\dfrac{1}{r^2}\dfrac{\partial^2 \upsilon_z}{\partial \theta^2}+\\ +\dfrac{\partial^2 \upsilon_z}{\partial z^2}+\dfrac{1}{r}\dfrac{\partial \upsilon_z}{\partial r}\end{pmatrix},\end{aligned} \tag{14}$$

Taking into account the assumptions made, regarding the axial symmetry of the flow, it is possible to assume $\vec{F}\equiv\vec{0},\upsilon_\theta\equiv 0,\frac{\partial}{\partial\theta}\equiv 0$. Then equations (11)-(14) will take the form:

$$\frac{\partial \upsilon_r}{\partial r}+\frac{\partial \upsilon_z}{\partial z}+\frac{\upsilon_r}{r}=0 \tag{15}$$

$$\begin{aligned}&\frac{\partial \upsilon_r}{\partial t}+\upsilon_r\frac{\partial \upsilon_r}{\partial r}+\upsilon_z\frac{\partial \upsilon_r}{\partial z}=\\ &=\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\Big(\frac{\partial^2 \upsilon_r}{\partial r^2}+\frac{\partial^2 \upsilon_r}{\partial z^2}+\frac{1}{r}\frac{\partial \upsilon_r}{\partial r}-\frac{\upsilon_r}{r^2}\Big),\end{aligned} \tag{16}$$

$$\begin{aligned}&\frac{\partial \upsilon_z}{\partial t}+\upsilon_r\frac{\partial \upsilon_z}{\partial r}+\upsilon_z\frac{\partial \upsilon_z}{\partial z}=\\ &=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\Big(\frac{\partial^2 \upsilon_z}{\partial r^2}+\frac{\partial^2 \upsilon_z}{\partial z^2}+\frac{1}{r}\frac{\partial \upsilon_z}{\partial r}\Big).\end{aligned} \tag{17}$$

Remark 1. The assumption $\upsilon_\theta\equiv 0$, in general, is not critical: it is enough to assume that $\upsilon_\theta=\upsilon_\theta(t,r,z)$. In this case one mote equation will be added to equations (15)-(17):

$$\begin{aligned}&\frac{\partial \upsilon_\theta}{\partial t}+\upsilon_r\frac{\partial \upsilon_r}{\partial r}+\upsilon_z\frac{\partial \upsilon_\theta}{\partial z}+\frac{\upsilon_r\upsilon_\theta}{r}=\\ &=\nu\Big(\frac{\partial^2 \upsilon_\theta}{\partial r^2}+\frac{\partial^2 \upsilon_\theta}{\partial z^2}+\frac{1}{r}\frac{\partial \upsilon_\theta}{\partial r}-\frac{\upsilon_\theta}{r^2}\Big).\end{aligned} \tag{18}$$

The system, however, remains closed: the number of unknown functions coincides with the number of equations.

Remark 2. When solving (in particular, numerically) partial differential equations (system of equations), it is important to investigate the correctness of the corresponding initial-boundary problems for the equations being solved. Such research is very nontrivial, especially for quasilinear partial differential equations of the second order, such as Navier-Stokes equations. For the equation system (11)-(14), fortunately, such investigation (with a positive result) was carried out by O.A. Ladyzhenskaya [7].

3. *Numerical solution of the initial-boundary problem for the Navier-Stokes equation,*

In this paragraph we will give an example of the calculation of the parameters of the flow of the viscous incompressible liquid through a straight pipe of the circular cross-section, described by equation system (15)-(17). We define initial and boundary conditions from some reasonable physical formulation of the problem.

1. At the pipe inlet and outlet the flow is supposed to be laminar, i.e.

$$\upsilon_r(r,0,t)=\upsilon_r(r,L,t)=0\,. \tag{19}$$

2. At the initial moment of time the flow is laminar and steady, i.e.

$$\upsilon_r(r,z,0)=0,\ \upsilon_z(r,z,0)=\varphi(r). \tag{20}$$

As an initial velocity distribution $\varphi(r)$ we take the Poiseuille distribution (see [5]).

3. Because of the viscous drag force at the wall of pipe the condition of flow adhesion is fulfilled

$$\upsilon_r(R,z,t)=0,\ \upsilon_z(R,z,t)=0\,. \tag{21}$$

4. We suppose the pressure to be set at the pipe inlet and outlet

$$p(r,0,t)=p_1,\ p(r,L,t)=p_2\,. \tag{22}$$

5. The condition of zero pressure gradient must be fulfilled at the wall of the pipe

$$\left.\Big(\frac{\partial p}{\partial r}+\frac{\partial p}{\partial z}\Big)\right|_{r=R}=0\,. \tag{23}$$

We shall seek the numerical solution of the problem (15)-(17), (19)-(23), prior reduced to the dimensionless form. We use the alternating direction method [8]. According to this method we will find the velocity projections from the following difference equations:

$$\begin{aligned}&\frac{u_{i,j}^{k+0.5}-u_{i,j}^{k}}{0.5\cdot h_\tau}=\frac{u_{i+1,j}^{k+0.5}-2u_{i,j}^{k+0.5}+u_{i-1,j}^{k+0.5}}{h_\eta^2}+\frac{1}{\eta_i}\frac{u_{i+1,j}^{k+0.5}-u_{i-1,j}^{k+0.5}}{2h_\eta}-\\ &-\frac{u_{i,j}^{k}}{\eta_i^2}+\frac{u_{i,j+1}^{k}-2u_{i,j}^{k}+u_{i,j-1}^{k}}{h_\xi^2}-\frac{u_{i,j}^{k}+\left|u_{i,j}^{k}\right|}{2}\frac{u_{i+1,j}^{k+0.5}-u_{i,j}^{k+0.5}}{h_\eta}-\\ &-\frac{u_{i,j}^{k}-\left|u_{i,j}^{k}\right|}{2}\frac{u_{i,j}^{k+0.5}-u_{i-1,j}^{k+0.5}}{h_\eta}-\frac{w_{i,j}^{k}+\left|w_{i,j}^{k}\right|}{2}\frac{u_{i,j+1}^{k}-u_{i,j}^{k}}{h_\xi}-\\ &-\frac{w_{i,j}^{k}-\left|w_{i,j}^{k}\right|}{2}\frac{u_{i,j}^{k}-u_{i,j-1}^{k}}{h_\xi},\end{aligned} \tag{24}$$

$$\begin{aligned}&\frac{u_{i,j}^{k+1}-u_{i,j}^{k+0.5}}{0.5\cdot h_\tau}=\frac{u_{i+1,j}^{k+0.5}-2u_{i,j}^{k+0.5}+u_{i-1,j}^{k+0.5}}{h_\eta^2}+\frac{1}{\eta_i}\frac{u_{i+1,j}^{k+0.5}-u_{i-1,j}^{k+0.5}}{2h_\eta}-\\ &-\frac{u_{i,j}^{k+0.5}}{\eta_i^2}+\frac{u_{i,j+1}^{k+1}-2u_{i,j}^{k+1}+u_{i,j-1}^{k+1}}{h_\xi^2}-\frac{u_{i,j}^{k+0.5}+\left|u_{i,j}^{k+0.5}\right|}{2}\frac{u_{i+1,j}^{k+0.5}-u_{i,j}^{k+0.5}}{h_\eta}-\\ &-\frac{u_{i,j}^{k+0.5}-\left|u_{i,j}^{k+0.5}\right|}{2}\frac{u_{i,j}^{k+0.5}-u_{i-1,j}^{k+0.5}}{h_\eta}-\frac{w_{i,j}^{k+0.5}+\left|w_{i,j}^{k+0.5}\right|}{2}\frac{u_{i,j+1}^{k+1}-u_{i,j}^{k+1}}{h_\xi}-\\ &-\frac{w_{i,j}^{k+0.5}-\left|w_{i,j}^{k+0.5}\right|}{2}\frac{u_{i,j}^{k+1}-u_{i,j-1}^{k+1}}{h_\xi}-\frac{\gamma_{i+1,j+1}^{k+1}+\gamma_{i+1,j-1}^{k+1}-\gamma_{i-1,j+1}^{k+1}-\gamma_{i-1,j-1}^{k+1}}{2h_\eta+2h_\xi},\end{aligned} \tag{25}$$

$$\frac{w_{i,j}^{k+0.5}-w_{i,j}^{k}}{0.5\cdot h_\tau}=\frac{w_{i+1,j}^{k+0.5}-2w_{i,j}^{k+0.5}+w_{i-1,j}^{k+0.5}}{h_\eta^2}+\frac{1}{\eta_i}\frac{w_{i+1,j}^{k+0.5}-w_{i-1,j}^{k+0.5}}{2h_\eta}+$$
$$+\frac{w_{i,j+1}^{k}-2w_{i,j}^{k}+w_{i,j-1}^{k}}{h_\xi^2}-\frac{u_{i,j}^{k}+\left|u_{i,j}^{k}\right|}{2}\frac{w_{i+1,j}^{k+0.5}-w_{i,j}^{k+0.5}}{h_\eta}- \quad (26)$$
$$-\frac{u_{i,j}^{k}-\left|u_{i,j}^{k}\right|}{2}\frac{w_{i,j}^{k+0.5}-w_{i-1,j}^{k+0.5}}{h_\eta}-\frac{w_{i,j}^{k}+\left|w_{i,j}^{k}\right|}{2}\frac{w_{i,j+1}^{k}-w_{i,j}^{k}}{h_\xi}-$$
$$-\frac{w_{i,j}^{k}-\left|w_{i,j}^{k}\right|}{2}\frac{w_{i,j}^{k}-w_{i,j-1}^{k}}{h_\xi},$$

$$\frac{w_{i,j}^{k+1}-w_{i,j}^{k+0.5}}{0.5\cdot h_\tau}=\frac{w_{i+1,j}^{k+0.5}-2w_{i,j}^{k+0.5}+w_{i-1,j}^{k+0.5}}{h_\eta^2}+\frac{1}{\eta_i}\frac{w_{i+1,j}^{k+0.5}-w_{i-1,j}^{k+0.5}}{2h_\eta}+$$
$$+\frac{w_{i,j+1}^{k+1}-2w_{i,j}^{k+1}+w_{i,j-1}^{k+1}}{h_\xi^2}-\frac{u_{i,j}^{k+0.5}+\left|u_{i,j}^{k+0.5}\right|}{2}\frac{w_{i+1,j}^{k+0.5}-w_{i,j}^{k+0.5}}{h_\eta}- \quad (27)$$
$$-\frac{u_{i,j}^{k+0.5}-\left|u_{i,j}^{k+0.5}\right|}{2}\frac{w_{i,j}^{k+0.5}-w_{i-1,j}^{k+0.5}}{h_\eta}-\frac{w_{i,j}^{k+0.5}+\left|w_{i,j}^{k+0.5}\right|}{2}\frac{w_{i,j+1}^{k+1}-w_{i,j}^{k+1}}{h_\xi}-$$
$$-\frac{w_{i,j}^{k+0.5}-\left|w_{i,j}^{k+0.5}\right|}{2}\frac{w_{i,j}^{k+1}-w_{i,j-1}^{k+1}}{h_\xi}-\frac{\gamma_{i+1,j+1}^{k+1}+\gamma_{i+1,j-1}^{k+1}-\gamma_{i-1,j+1}^{k+1}-\gamma_{i-1,j-1}^{k+1}}{2h_\eta+2h_\xi},$$

where u, w are dimensionless velocity projections υ_r and υ_z respectively, η,ξ,τ are dimensionless radial coordinate, linear coordinate and time respectively, h_η, h_ξ, h_τ are steps along the corresponding coordinate axes. These equations are reduced to the form $Ax_{i-1}+Bx_i+Cx_{i+1}=F_i$ and are solved with the tridiagonal matrix algorithm.
We obtain the equation for determining the pressure by differentiating equations (16) and (17) with respect to r and z, and adding them:

$$\Delta p=\rho\left\{\begin{aligned}&\nu\left(\frac{\partial^2 D}{\partial r^2}+\frac{1}{r}\frac{\partial D}{\partial r}+\frac{\partial^2 D}{\partial z^2}\right)-\frac{\partial D}{\partial t}-u\frac{\partial D}{\partial r}-\\&-w\frac{\partial D}{\partial z}-\left[\frac{\partial u}{\partial r}\right]^2-\left[\frac{\partial w}{\partial z}\right]^2-2\frac{\partial w}{\partial r}\frac{\partial u}{\partial z}\end{aligned}\right\}, \quad (28)$$

where $D=\frac{\partial\upsilon_r}{\partial r}+\frac{\partial\upsilon_z}{\partial z}+\frac{\upsilon_r}{r}$. For the equation (28) we also write the difference approximation. The pressure (or rather γ – the dimensionless function corresponding to the relation p/ρ) we shall seek with the Richardson method [8]:

$$\gamma_{i,j}^{s+1}=F\left(\gamma_{i+1,j}^{s},\gamma_{i-1,j}^{s},\gamma_{i,j+1}^{s},\gamma_{i,j-1}^{s}\right) \quad (29)$$

where s denotes the iteration number. The computation by formula (27) continues, while the difference between values in two adjacent iterations doesn't become less than the preset accuracy ε. Obviously, the equations given have a singularity at the point $r=0$. This obstacle can be bypassed, if we assume $\Delta_r(f)=2\frac{\partial^2 f}{\partial r^2}$ when $r=0$.

The formulas given were implemented as a program in the Matlab environment. The results are shown in the form of graphics in Fig.1,2,3 and 4.

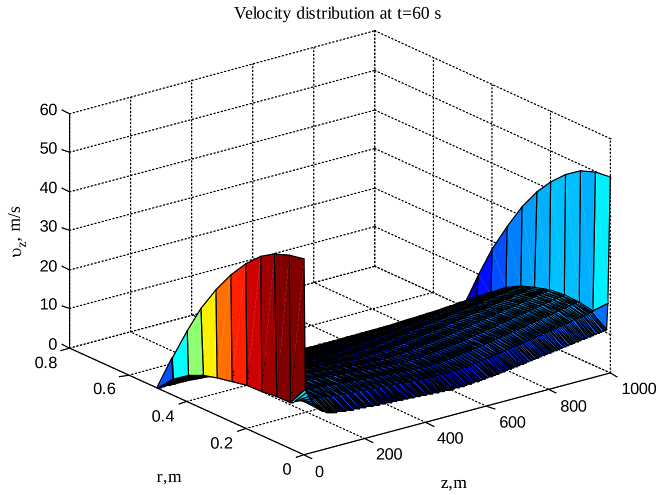

Fig. 1. The velocity υ_z field

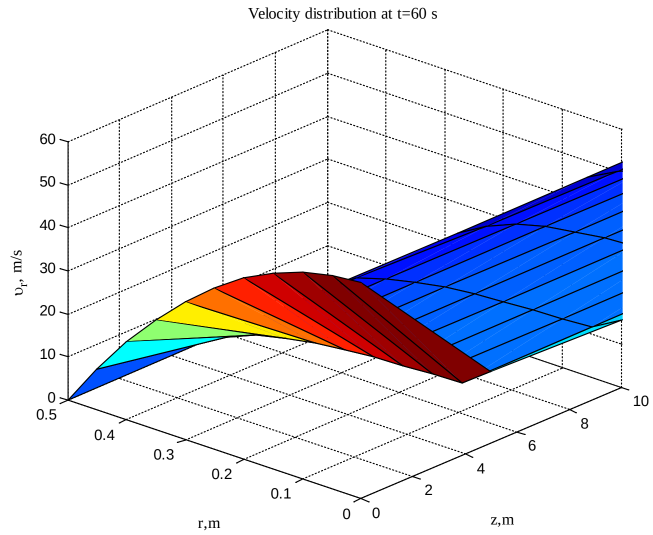

Fig. 2. The velocity υ_z field on the segment $z\in[0,10]$

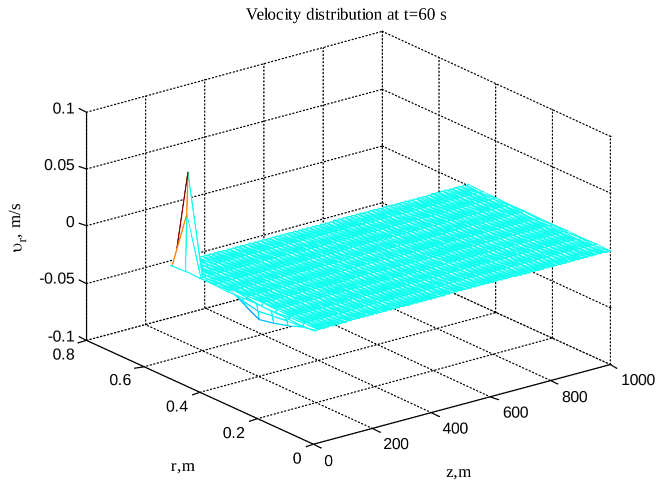

Fig. 3. The velocity υ_r field

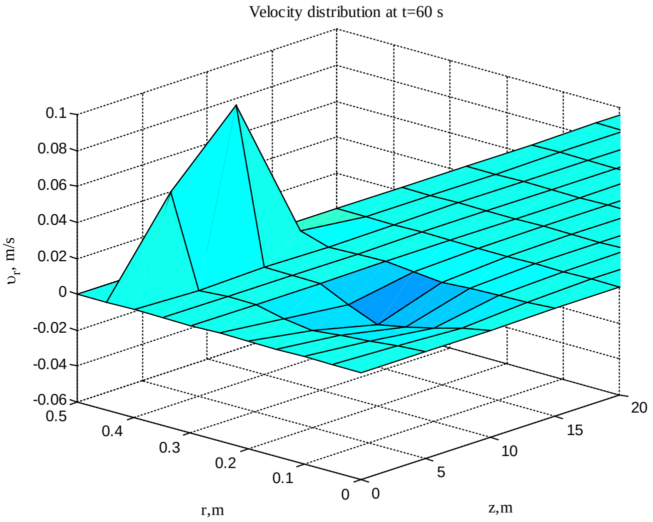

Fig. 4. The velocity υ_r field on the segment $z \in [0,10]$

4. Conclusion

In this paper the mathematical model of the real viscous liquids motion along the horizontal cylindrical pipe is proposed and the implicit difference scheme is built for solving corresponding initial-boundary problems. The scheme was tested on a typical example.

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