THE MATHEMATICAL MODEL AS A BASIS FOR THE NATURAL CLASSIFICATION OF SYSTEMS AND PROCESSES

МАТЕМАТИЧЕСКАЯ МОДЕЛЬ КАК ОСНОВА ЕСТЕСТВЕННОЙ КЛАССИФИКАЦИИ СИСТЕМ И ПРОЦЕССОВ

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Abstract: *this paper attempts to provide a transparent and, if possible, a formal hierarchy of the main types of mathematical models used in the description of dynamic processes inside, at first glance, different systems. It is emphasized that the mathematical modeling is a natural and universal environment for effective analysis of system processes of different nature. From our point of view, the term "system analysis" means the methodology for classification of real systems (physical, biological, economic, social, etc.), which is based on the classification of mathematical models that are used to describe these systems.*
***KEYWORDS**: SYSTEMS, SYSTEM ANALYSIS, MATHEMATICAL MODELS, MATHEMATICAL MODELING, CLASSIFICATION*

1. Introduction

We define a system as a collection (set) of objects[1] of a particular nature, interacting with each other and the external environment. Since every interaction presupposes the possibility of state changes as the objects themselves that make up the system, and the system as a whole, the first task of the researcher is to establish the patterns of these changes in order to predict the latter. The second task is the task of forming effects on the system as a whole (through changes in environmental conditions) or of its component objects to implement the required changes to the characteristics of the system and/or its objects. In other words, we have to talk about the study and management of the processes occurring in the system.

For definiteness, formulations, point to a sense that we put here in the term "characteristics". Under the characteristics of the system, we understand the qualitative or functional description important for the purposes of the study of the properties of the system as a whole and its constituent objects. A set of important characteristics are not cast in stone, and can vary in research depending on the results obtained, change the purposes of the study, further information (experiments, calculations, discoveries, etc.). Here are a few examples. In classical mechanics the important characteristics of the free material point are its coordinates (more precisely, the radius vector) and momentum. For holonomic system of material points – a set of generalized coordinates and velocities (momenta) of the system and the Lagrangian function[2], the latter, as you know, is a function of generalized coordinates and generalized speeds of the system and, possibly, time. In quantum mechanics, the important characteristics for the particles are, for example, the wave function, the energy (operator) and spin (number). In gas dynamics, the important characteristics are functions of the spatial coordinates and time (density, velocity, pressure, temperature, etc.). In classical molecular-kinetic theory of gases basic properties of a system of molecules are the distribution function and the law of interaction of molecules between themselves, representing the operator.

2. On the modeling of systems

In practice, the learning processes in any real system, is reduced to their modeling i.e. to approximate description by creating their simplified models. Traditionally, models of systems and processes are dividing into two main classes: physical and mathematical models, although this division is largely arbitrary, since, as we shall see below, the physical model without its mathematical counterpart loses much of its effectiveness and practical significance. These models should go "hand in hand" in serious research.

Under a *physical model*, we usually mean a product, device or imaginary system, which simplified the likeness of the object or system, and allows you to recreate the monitoring process or phenomenon with the required degree of accuracy. At the same time, elements of the system are associated with the physical equivalents, reproducing the structure, basic properties and relations of the studied object or process. Typically, the physical model of the object or system should have the same qualitative nature as the simulated object (system). The basis of constructing an adequate physical model are placed, as a rule, the idea of the theory of similarity and dimension.

Under a *mathematical model*, we usually mean an approximate description of a class of phenomena of the external world, expressed by mathematical symbols. Most often, the mathematical model is a set of equations and/or inequalities (algebraic, differential, integral, operator, functional). Arsenal research methods of mathematical models is extremely wide - from the logic of algebra to functional analysis[3]. In this case, since the mathematical model of the object is "mathematical world", it should be investigated on the received in this "world" level of rigor. Finally, it is important to understand that a mathematical model cannot "be proved". It is always a hypothesis, whose validity is checked only by its practical application.

3. Stages and features of mathematical modeling

The process of mathematical modeling, i.e. studying phenomena of any nature with the help of a mathematical model, can be divided into four stages [1].

The first phase - the formulation of laws connecting the main objects of the model. This stage requires a broad knowledge of the facts relating to the phenomenon, and deep penetration into their relationship. This stage is completed in mathematical terms formulated qualitative ideas about the links between model objects.

The second stage - the study of mathematical problems, which leads to a mathematical model. The main issue here is the solution of the direct problem, i.e., obtaining as a result of the analysis of model output (theoretical consequences) for further comparison with the results of observations of the phenomena studied. At this stage, the mathematical apparatus acquires the main role that is necessary for the analysis of mathematical models and computer engineering - a powerful tool for obtaining quantitative output data as a result of solving complex mathematical problems.

The third stage - clarification as to whether the adopted (hypothetical) model is the criterion of practice, i.e., asking the

[1] The objects that make up the system, often referred to as its elements.
[2] On the other hand, the total energy of the system.
[3] Including, of course, numerical methods. However, the method of their application and interpretation of the results must be mathematically strictly justified.

question of whether the results are consistent with the observations of the theoretical consequences of the model within the observational accuracy. If deviations are outside the limits of accuracy of the observations, the model cannot be accepted. It often happens in the construction of the model that some of its characteristics remain uncertain. Tasks, which define the characteristics of the model (parametric and functional) so that the output information was comparable within the accuracy of the observations with observations of the phenomena studied, called inverse problems. If such compatibility cannot be achieved for any choice of the characteristics, the model is not suitable for the study of the phenomena under consideration. The use of practical criterion to appraise a mathematical model allows to conclude that the validity of the assumptions underlying the study of the subject (hypothetical) model.

The fourth stage - the subsequent analysis of the model due to the accumulation of data on the phenomenon, and the model of modernization. In the development of science and technology, phenomenological data are more and more refined, and there comes a point of time when the conclusions derived on the basis of an adopted mathematical model do not correspond to our knowledge of the phenomenon. Thus, there is a need to build a new, more sophisticated mathematical model.

A good illustration of the above can be an example of the development of the theory of gravitation. The classical theory of gravitation Newton's mathematical point of view was based on Euclidean space-time model that eventually led to the description of the dynamics of the system of material bodies based on the Lagrange equations. Experience has shown that the Newton-Lagrange theory adequately describes the mechanical processes that we face on Earth and in space within the solar system. A brilliant example of this - the discovery of the planet Neptune, which was made in 1846 by German astronomer Johann Galle based on calculations, made by Urbain Le Verrier (France) and John Adams (England), is based on Newton's theory of gravitation. However, the development of cosmological studies in the early twentieth century led to the discovery of phenomena that could not be explained in the framework of the classical theory of gravitation. The search for answers to your questions led Einstein (in 1916) to construct a new mathematical model of space-time, which was the basis for the hypothesis of space-time as a Riemann space, in which the curvature tensor depends on the distribution and magnitude of the masses of space objects. In this space, in particular, light (more precisely, the photons) extends not straight (as is the Euclidean model), and on geodesic lines corresponding Riemann space. Einstein's theory was confirmed experimentally in 1919, when there was a ray of light deviation from the star as it passes close to the sun that qualitatively and quantitatively consistent with the predictions of the theory. On the other hand, in the absence of the objects near the large (on space standards) gravitating mass, corrections (with respect to the classical theory of gravitation), given by Einstein's theory, are, from a practical point of view, negligible. This allows you to successfully continue to use the relevant conditions of Newton's theory of gravitation.

A.N. Tikhonov in the article in «Mathematical encyclopedia» cited above gives another example illustrating the steps of mathematical modeling.

Concluding the topic of physical and mathematical models point to stand somewhat apart the analog model. This type of model clearly demonstrates the organic link between methods of physical and mathematical modeling. The essence of the analog simulation is as follows. Consider two systems of different physical nature (e.g., mechanical and electrical). Processes in these systems proceed according to qualitatively different physical laws. If, however, the mathematical models of both systems have the same form, i.e., from a mathematical point of view, these systems are the same, based on the study of the behavior of a system can be inferred about the behavior of another system[4].

As an example, consider the following mechanical and electrical systems [2]. Let the mechanical system with one degree of freedom is a mass m, moving under the action of which the exciting force p, elastic restoring force (e.g., spring), characterized by stiffness c, and the damping force of viscous friction with coefficient r. Let h - coordinate of the mass m, which determines its deviation from its equilibrium position. The equation of motion of a mechanical system has the form

$$m\ddot{h} + r\dot{h} + ch = p. \qquad (1)$$

Suppose further that the electrical system is an electrical circuit consisting of series-connected source voltage (e), the inductance of L, resistance R, and the inverted container S ($S = 1 / C$). Let q - the electric charge transferred by a current i through the conductor cross-section:

$$q = q_0 + \int_0^t i dt.$$

The second Kirchhoff's law leads to the following equation:

$$L\ddot{q} + R\dot{q} + Sq = e. \qquad (2)$$

If the values of p and e are changed as a function of one and the same law, the equations (1) and (2) are substantially identical and, therefore, to study the behavior of the mechanical system can be replaced by studying the electrical behavior of the system and vice versa. Thus, these two systems in terms of their mathematical models are similar to each other, although they are different systems of physical nature.

The above examples touch simulation "material and physical" systems, but all of the above applies equally to problems of mathematical modeling of systems of different nature - social, economic, etc. Let us give an example. According to T. Parsons (see [3], ch. 9), four numerical functions, depending on the time t, can be assigned to the socio-economic system, which sufficiently adequately describe the state of this system: $G(t)$, characterizing the political system in society; $E(t)$, characterizing the economic system; $K(t)$, which characterizes the social community; and $D(t)$, which characterizes the system of maintaining institutional ethnic images. Parsons showed that these functions must satisfy some autonomous system of differential equations of the form:

$$\frac{dG}{dt} = F_1(G,E,K,D); \quad \frac{dE}{dt} = F_2(G,E,K,D);$$

$$\frac{dK}{dt} = F_3(G,E,K,D); \quad \frac{dD}{dt} = F_4(G,E,K,D),$$

where the functions F_j are given sufficiently smooth functions of their arguments. Thus, for this system, for example, the Lyapunov theorem on stability in the first approximation is valid. This, in turn, allows us to formulate certain conditions for the parameters included in the expressions for F_j, which guarantee a stable dynamics of the social community.

In other words, the behavior of systems "made of different test" can be investigated effectively by using models built in one of them - mathematical. It is only necessary to select the appropriate mathematical apparatus. For example, for the construction and study of adequate mathematical models of social and economic processes have to resort to such seemingly quite abstract, analytical areas of mathematics as catastrophe theory and algebraic topology (see., e.g., [3, 4]).

4. *System analysis and adequacy of mathematical models*

The foregoing gives grounds to propose to classify the system (regardless of their actual nature) according to the classes of mathematical models used to describe these systems.

[4] It is necessary to distinguish between analogue and similar models studied in the theory of similarity and dimension. In the latter case, we are dealing with models of the same nature, while the analogue model is a model of a different nature.

Thus, it seems reasonable to take as a basis the following definitions.

System is a collection of objects highlighted by a researcher, which are interacting with each other and the environment.

Systems analysis is a methodology for classification of real (physical, technical, biological, economic, social, etc.) systems, based on the classification of mathematical models used for formalized description of these systems.

That is, we unite into the same class those real systems, formalized description of which requires the use of the same class of mathematical models (e.g., ordinary differential equations, partial differential equations, operator equations in function spaces, probability theory, mathematical logic, graph theory, etc.).

Speaking about the importance of mathematical modeling in the analysis of processes in different systems, it is necessary to say a few words about the features of this simulation. If we want to use mathematical methods for solving problems related to any technical, economic, biological or any other system, we must, as mentioned above, to start with the construction of a mathematical model of the relevant system. Build a model - which means indicate what mathematical objects under study is characterized by the system and how these mathematical concepts reflect the laws that describe the functioning of the system.

Since the mathematical model cannot display the studied reality in all its concreteness and fullness, when its construction is necessary to choose such factors and the relationships that are most important in this problem. In many cases, the right to choose the model - means to solve the problem by more than half. The main difficulty in this case lies in the fact that the construction of the model requires special connection (for the study of the qualitative features of the simulated process) and mathematical knowledge.

When constructing a mathematical model can meet and insurmountable (at the moment) the difficulty, for example,

– Insufficient knowledge of laws that describe the system (for example, biology, economics, sociology);

– We do not know some significant characteristics of the system (for example, the density distribution of matter in the interior of the Earth, the pressure in the blood vessels of a living organism).

In such cases, the construction of the model is necessary to make additional assumptions that have the character of hypotheses. Consequently, the conclusions drawn from the study of this model are conditional. But even if the laws are known and measured characteristics of the system, the model can get so complicated that it would be available for the study of well-known (for now) with mathematical methods.

Therefore, we often have to make some simplifications in the simulation of the real process, idealized model.

Thus, the construction of a mathematical model must meet two important criteria:

1) the model must be adequate, i.e. to the extent necessary correspond to reality.

2) The model must be simple enough to allow finding an effective solution within a reasonable time and with reasonable accuracy.

It is easy to note that these criteria are contradictory. On the one hand, to build adequate models requires consideration of more complete and varied number of factors, which inevitably complicates the model. This complexity leads to that the posed problem cannot be solved at all or for practical time. On the other hand, over-simple model, which does not take into account a number of important factors, is clearly inadequate, and therefore does not meet its intended purpose. Art researcher is to find a harmonious matching of these two mutually exclusive requirements. We should seek to ensure that the model is consistent with reality, other things being equal. However, the degree of compliance must be determined by the terms of the problem, i.e., the model should provide enough (within the required accuracy) true reflection of reality.

In connection with the above, you can specify the steps of the solution of applied problem using mathematical methods:

1. Correct application wording task is to extract the essential properties and relationships of the studied object or process.

2. Construction of mathematical model - display of quantitative (functional) relationships allocated for the first stage of the properties with the help of mathematical concepts and relationships. As noted above, this is the most difficult part of the study. Firstly, for the same phenomena of a variety of mathematical models can be built. Secondly, it is necessary to satisfy the requirements of the constructed model adequacy and simplicity.

3. Logical-mathematical analysis of the model: model checking for consistency, and the solution of a mathematical problem. Sometimes the implementation of this phase requires the development of new methods or even creating a new mathematical discipline. The solution of the mathematical problem cannot always be found for a particular formula, even if it exists. Therefore may require numerical solution methods, - i.e., methods which give no exact and approximate (with reasonable accuracy) response. This phase includes the theoretical study of mathematical problem and its practical solution (including baseline data collection, programming).

4. Interpretation of results from a known empirical data (the results of observations or experiments specially designed), i.e. translating mathematical answer to the specific language of science.

5. Concluding remarks

In conclusion, I would like to note the following. Due to the increasing amount of scientific information and the deepening of the process of differentiation of sciences one researcher has become almost impossible to work in several branches of knowledge. Therefore, in order to construct adequate models and effectively solve mathematical problems arising in connection with this (especially when it comes to major scientific problems), requires the cooperation of scientists from different specialties and, of course, the attraction of modern computer technology. Hence it follows that even a good specialist in a certain field of knowledge can now be unarmed against the problems arising in the investigation if he is not a kind of "synthesizer of knowledge": he must well imagine what and where is located in science. Only such a person can intelligently choose the appropriate research team and direct it to work in the right direction. Training and education of such specialists is not an easy, but extremely important problem, facing the system of higher education.

Literature

1. A. N. Tikhonov. Mathematical model. - In the book “Mathematical encyclopedia” in 5 volumes, vol. 3. - M.: Publishing house "Soviet Encyclopedia", 1982 (in Russian).
2. A.Yu. L’vovich. Electromechanical systems: Textbook. - L.: Publishing House of Leningrad Univ., 1989 (in Russian).
3. A.K. Guts, et al. Mathematical Models of Social Systems: Textbook – Omsk: Omsk State University, 2000 (in Russian).
4. Problems of optimization and economic applications: reports of the V All-Russian Conference (Omsk, 2 - 6 July 2012). - Omsk: Omsk State University, 2012 (in Russian).
5. I.I. Blekhman, A.D. Myshkis, Ya.G. Panovko. Applied Mathematics. The subject, logics, features of approaches. - Kiev, "Naukova Dumka", 1976 (in Russian).
6. I.I. Blekhman, A.D. Myshkis, Ya.G. Panovko. Mechanics and applied mathematics: logic and application features. - M.: “Nauka”, 1990 (in Russian).