EXACT RECONSTRUCTION VERSION OF RADON TRANSFORMATION IN TOMOSYNTHESIS

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Abstract: *The purpose of this study is a method of exact reconstruction in the Radon problem, which consists in refusal of the approximate transformation kernel usage. A comparison of the methods that are currently used in tomosynthesis was conducted. Model experiments were performed; the results of the application of the proposed method in real tomography studies in tomography are given.*

The traditional methods of Radon transformation [1] do not make it possible to accurately reconstruct the initial distribution function due to the divergence of the integral expression. In this paper, we suggest alternative method aiming to get rid of this defect by changing the order of integration using re-grouping the integrand functions in the result of transformation:

$$f(x,y)=\int\frac{|Q|dQd\alpha dX}{(2\pi)}F(X,\alpha)\exp(-iQ(X-x\sin\alpha+y\cos\alpha)). \qquad (1)$$

Here $f(x,y)$ is the desired density distribution in the layer, $F(X,\alpha)$ is the projections made under angle α, and X is the coordinate in the detector. Usual way to use (1) is introduce function (filter)

$$G(z)=\int\frac{|Q|dQ}{(2\pi)}\exp(-iQz)=\frac{1}{\pi}\int_0^\infty|Q|\cos(Qz)dz\,, \qquad (2)$$

which describes the connection between $f(x,y)$ and $F(X,\alpha)$:

$$f(x,y)=\int d\alpha dXF(X,\alpha)G(X-x\sin\alpha+y\cos\alpha). \qquad (3)$$

The main problem here is that the integral (2) contains a singularity and, therefore, approximate functions [2] are used instead of it, which reflect the main features of the function (2), but lead to an approximate recovery.

In this paper, we propose to change the order of integration to avoid this problem. In this case none of the integrals into this expression (1) diverges within the limits. As a result of taking the integral in this way, we eventually have to obtain the distribution function equivalent to the original one.

Let’s consider the case where the initial distribution function is the delta function, which is determined by the following expression

$$f(x,y)=\rho_0 a^2\delta(x-x_0)(y-y_0) \qquad (4)$$

which corresponds to the projection:

$$F(X,\alpha)=\rho_0 a^2\delta(x_0\sin(\alpha)-y_0\cos(\alpha)-X) \qquad (5)$$

Whatever divergences arise in the integral expression (1), we shall not get it to the form containing the kernel of the Radon transformations (2) in explicit form, but immediately substitute the expression (5) into it. Thus, the relation (1), taking into account the integration limits, can be written in the form:

$$f(x,y)=\frac{\rho_0 a^2}{(2\pi)^2}\int_{-\infty}^{+\infty}QdQ\int_0^\pi d\alpha\int_{-\infty}^{+\infty}dX\,\delta(x_0\sin(\alpha)-y_0\cos(\alpha)-X)\,*$$
$$e^{-iQX}e^{iQ(x\sin(\alpha)-y\cos(\alpha))} \qquad (6)$$

Now integrating this expression first with dX, getting rid of the delta functions. As a result, we get:

$$f(x,y)=\frac{\rho_0 a^2}{(2\pi)^2}\int_{-\infty}^{+\infty}QdQ\int_0^\pi d\alpha\;\; e^{iQ((x-x_0)\sin(\alpha)-(y-y_0)\cos(\alpha))} \qquad (7)$$

Now we return to other integration variables to avoid divergence and make integration possible in a different order. It is logical to go over to the variables $(\tilde{k}_x,\tilde{k}_y)$, for this we write the Jacobian of the transition:

$$dQd\alpha = \begin{vmatrix} \widetilde{k_x} = Q\sin(\alpha) \\ \widetilde{k_y} = Q\cos(\alpha) \end{vmatrix} = d\tilde{k}_x d\tilde{k}_y \frac{\partial(Q,\alpha)}{\partial(k_x,k_y)} = \frac{d\tilde{k}_x d\tilde{k}_y}{Q}, \tag{8}$$

taking this into account, expression (7) can be written as follows:

$$f(x,y) = \frac{\rho_0 a^2}{(2\pi)^2}\int_{-0}^{+\infty} e^{i\widetilde{k_x}(x-x_0)} d\tilde{k}_x \int_0^{\infty} e^{-i\widetilde{k_y}(y-y_0)}\, d\tilde{k}_y. \tag{9}$$

It is easy to see that the two integrals in the expression are just the Fourier representation of the delta function, similar to the representation:

$$\delta(x-x_1) = \int \frac{dQ}{2\pi} e^{-iQ(x-x_1)} \tag{10}$$

Thus, we get:

$$f(x,y) = \frac{\rho_0 a^2}{(2\pi)^2}\, 2\pi\, \delta(x-x_0)\, 2\pi\, \delta(y-y_0) \tag{11}$$

Hence, we get the answer:

$$(x,y) = \rho_0 a^2\, \delta(x-x_0)\ \delta(y-y_0) \tag{12}$$

We can see that this function completely identical to the function (4) introduced by us, moreover, we had no need to enter redefinitions anywhere, since all the transitions were initially equal. As a result, we showed that the delta distribution function can be accurately reconstructed by the proposed method.

After the prove above that the delta function can be accurately reconstructed, natural to assume that any other function can also be reconstructed without loss of precision. We can prove it by taking into account that any distribution function can be represented as a continuous set of delta functions, and all the resulting interim expressions are additive quantities, which gives the right to put the sum or integral sign before the reconstructed distribution function.

This explanation can be considered as self-explanatory. Nevertheless, we strictly prove this statement without referring to the result already obtained. To do this, we return to the previously obtained formula (1) and change the order of integration:

$$f(x,y) = \frac{1}{(2\pi)^2}\int_0^{\infty} |Q|dQ \int_0^{\pi} d\alpha \int_{-\infty}^{+\infty} e^{-iQX}\, e^{iQx\sin(\alpha) - iQy\cos(\alpha)} F(X,\alpha)dX, \tag{13}$$

now the integration over dQ will be performed last.

To converge this integral, we recall the linear distribution

$$F(X,\alpha) = \int f(x,y)dl, \tag{14}$$

and represent $f(x,y)$ through the delta function as follows:

$$f(x,y) = \int\int d\tilde{x}d\tilde{y}\, \delta(x-\tilde{x})\delta(y-\tilde{y})f(\tilde{x},\tilde{y}), \tag{15}$$

Then we use the following presentations

$$F(X,\alpha) = \int dx\, dy\, \delta(x\sin(\alpha) - y\cos(\alpha) - X) f(x,y) \tag{16}$$

$$dl = dx\, dy\, \delta(x\sin(\alpha) - y\cos(\alpha) - X) \tag{17}$$

Putting this into (13) we get:

$$(x,y) = \frac{1}{(2\pi)^2}\int_0^{\infty} |Q|dQ \int_0^{\pi} d\alpha \int_{-\infty}^{+\infty} dX\, e^{-iQX}\, e^{iQx\sin(\alpha) - iQy\cos(\alpha)} \int\int\int\int d\tilde{x}d\tilde{y} * \delta(x-\tilde{x})\delta(y-\tilde{y})f(\tilde{x},\tilde{y})dx\, dy\, \delta(x\sin(\alpha) - y\cos(\alpha) - X) \tag{18}$$

Integrating by $dx\, dy$, we get:

$$f(x,y) = \frac{1}{(2\pi)^2}\int_0^{\infty} |Q|dQ \int_0^{\pi} d\alpha \int_{-\infty}^{+\infty} dX\, e^{-iQX}\, e^{iQx\sin(\alpha) - iQy\cos(\alpha)} \int\int d\tilde{x}d\tilde{y}\, f(\tilde{x},\tilde{y}) * \delta(\tilde{x}\sin(\alpha) - \tilde{y}\cos(\alpha) - X), \tag{19}$$

then integrate over dX and get:

$$f(x,y) =$$

$$= \frac{1}{(2\pi)^2} \int_0^\infty |Q| dQ \int_0^\pi d\alpha \; e^{iQx\sin(\alpha) - iQy\cos(\alpha)} \int \int e^{-iQ\tilde{x}\sin(\alpha) + iQ\tilde{y}\cos(\alpha)} f(\tilde{x},\tilde{y}) d\tilde{x} d\tilde{y} \quad (20)$$

Now we make the transition to new variables completely analogous to the transition (8), so we easily receive from (7) the expression:

$$f(x,y) = \frac{1}{(2\pi)^2} \int d\,k_x dk_y \; e^{ik_x(x-\tilde{x})} e^{ik_y(y-\tilde{y})} \int \int f(\tilde{x},\tilde{y}) d\tilde{x} d\tilde{y} \qquad (30)$$

Let’s consider the representation of the delta function again:

$$\delta(x - x_1) = \int \frac{dQ}{2\pi} e^{-iQ(x-x_1)} \qquad (31)$$

Applying it we get:

$$f(x,y) = \int \int d\tilde{x} d\tilde{y} \, \delta(x - \tilde{x}) \delta(y - \tilde{y}) f(\tilde{x},\tilde{y}) \qquad (32)$$

Comparing this result, we can see that this expression in fully equivalent to the expression (15).

Finally, we conclude that the proposed method absolutely allows reconstructing the initial distribution, regardless of what function it describes. Let us highlight, that this proof doesn’t contain any reference to the result of the reconstruction of the delta-like distribution function, so the proof done in the previous section can be considered as a particular case.

REFERENCES

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