EVOLUTIONARY MATHEMATICAL MODELS WITH DISTRIBUTED PARAMETERS ON THE NET AND NETLIKE DOMAIN

ЭВОЛЮЦИОННЫЕ МАТЕМАТИЧЕСКИЕ МОДЕЛИ С РАСПРЕДЕЛЕННЫМИ ПАРАМЕТРАМИ НА СЕТЯХ И СЕТЕПОДОБНЫХ ОБЛАСТЯХ

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Abstract: *Mathematical models of evolutionary processes on the network and setepodobnoj area. The method, which applies to many tasks of optimal control of differential systems, the status of which is defined by the weak solutions of evolutionary equations of mathematical physics on networks and setepodobnyh areas. This method is very common and is applicable to a wide class of linear tasks that have an interesting analogy with also multi-phase tasks of mechanics (in particular, the theory of plasticity). The results obtained in this manner for a specific equation with distributed parameters in the setepodobnoj area, serve not only to demonstrate the method, but also of interest to applications.*

Keywords: PARABOLIC EQUATION, DISTRIBUTED PARAMETERS, NETLIKE DOMAIN,WEAK SOLUTIONS, OPTIMAL CONTROL

1. Introduction

Mathematical models of evolutionary processes on the net and the netlike domain. The method, which applies to many tasks of optimal control of differential systems, the status of which is defined by the weak solutions of evolutionary equations of mathematical physics on the net and the netlike domain. This method is very common and is applicable to a wide class of linear tasks that have an interesting analogy with also multi-phase tasks of mechanics (in particular, the theory of plasticity). The results obtained in this manner for a specific equation with distributed parameters on the net or netlike domain, not only serves to demonstrate the method, but also of interest to applications.

2. Results and discussion

The study consists of two parts – an analysis of the evolution mathematical models with spatial variable that changes to the net (limited graf) and on the netlike domain. While each case is put and the optimal control problem is studied.

2.1. Mathematical model with distributed parameters on the net (graph)

We denote (see [1, 2]): Γ – limited-oriented geometric graph with edges γ which are parametrized by segment $[0,1]$; $\partial\Gamma$ and $J(\Gamma)$ – sets of boundary and internal nodes of the graph, respectively; Γ_0 – the union of all the edges that do not contain the endpoints; $\Gamma_t = \Gamma_0 \times (0,t)$, $\partial\Gamma_t = \partial\Gamma_0 \times (0,t)$ ($t \in [0,T]$).

Necessary space and sets: $C(\Gamma)$ – the space of continuous on Γ functions; $L_p(\Gamma)$ $(p = 1,2)$ --- Banach space measurable functions on the Γ_0, summable with a p-th degree (spaces $L_p(\Gamma_T)$ are defined similarly); $L_{2,1}(\Gamma_T)$ – the space of continuous from $L_1(\Gamma_T)$ with norm $\|u\|_{L_{2,1}(\Gamma_T)} = \int_0^T (\int_\Gamma u^2(x,t)dx)^{1/2} dt$; $W_2^1(\Gamma)$ – the space of continuous from $L_2(\Gamma)$, that have a generalized derivative of the 1st order also from the space $L_2(\Gamma)$; $W_2^{1,0}(\Gamma_T)$ – the space of continuous from $L_2(\Gamma_T)$, that have a generalized derivative of the 1st order of the variable x, that belongs $L_2(\Gamma_T)$ (similarly we introduce the space $W^1(\Gamma_T)$. Пусть $V_2(\Gamma_T)$ – the set of all functions $u(x,t) \in W_2^{1,0}(\Gamma_T)$ with finite norm $\|u\|_{2,\Gamma} = \max_{0\le t\le T} \|u(\cdot,t)\|_{L_2(\Gamma)} + \|u_x\|_{L_2(\Gamma)}$ and strongly continuous to t in norm $L_2(\Gamma)$.

We introduce the state space of parabolic systems and auxiliary space. For this we consider the bilinear form $\ell(\mu,\nu) = \int_\Gamma (a(x)\mu'(x)\nu'(x) + b(x)\mu(x)\nu(x))dx$ ($(\cdot)'$ – the generalized derivative) with fixed measurable and bounded on the Γ_0 functions $a(x)$, $b(x)$, square integrable. The space $W_2^1(\Gamma)$ includes a set $\Omega_a(\Gamma)$ of functions $u(x) \in C(\Gamma)$, satisfying the relations $\sum_{\gamma_j \in R(\xi)} a(1)_{\gamma_j} u'(1)_{\gamma_j} = \sum_{\gamma_j \in r(\xi)} a(0)_{\gamma_j} u'(0)_{\gamma_j}$ at all nodes $\xi \in J(\Gamma)$ ($R(\xi)$ and $r(\xi)$ – a plurality of edges, respectively oriented ``to the node $\xi$'' and ``from the node ξ'', $u(\cdot)_\gamma$ – restriction of the function $u(\cdot)$ on edge γ). Closure in norm $W_2^1(\Gamma)$ of the set $\Omega_a(\Gamma)$ will be denoted by

$W^1(a,\Gamma)$ (if we assume that the functions $u(x)$ from $\Omega_a(\Gamma)$ are also satisfy the boundary condition $u(x)\big|_{\partial\Gamma}=0$, we obtain a space $W_0^1(a,\Gamma)$). Suppose further $\Omega_a(\Gamma_T)$ – a variety of functions $u(x,t)\in V_2(\Gamma_T)$, whose traces are determined on section of the region Γ_T by the plane $t=t_0$ ($t_0\in[0,T]$) as function of class $W^1(a,\Gamma)$ and satisfy

(1) $$\sum_{\gamma_j\in R(\xi)} a(1)_{\gamma_j} u_x(1,t)_{\gamma_j} = \sum_{\gamma_j\in r(\xi)} a(0)_{\gamma_j} u_x(0,t)_{\gamma_j}$$

for all nodes $\xi\in J(\Gamma)$. The closure of the set $\Omega_a(\Gamma_T)$ in norm (1) let as $V^{1,0}(a,\Gamma_T)$; it's clear that $V^{1,0}(a,\Gamma_T)\subset W_2^{1,0}(\Gamma_T)$. Another subspace of the space $W_2^{1.0}(\Gamma_T)$ is $W^{1,0}(a,\Gamma_T)$ – closure in the norm $W_2^{1.0}(\Gamma_T)$ of the set of smooth functions satisfying (2) for all nodes $\xi\in J(\Gamma)$ and for all $t\in[0,T]$ (space $W^1(a,\Gamma_T)$ is entered similar); $V^{1,0}(a,\Gamma_T)$ – the state space of a parabolic system $W^{1,0}(a,\Gamma_T)$ and $W^1(a,\Gamma_T)$ – auxiliary spaces.

In the space $V^{1,0}(a,\Gamma_T)$ consider initial boundary value problem

(2) $$y_t(x,t)-(a(x)y_x(x,t))_x+b(x)y(x,t)= \\ = f(x,t),$$
$$y\big|_{t=0}=\psi(x),\quad x\in\Gamma,\quad a(x)y_x\big|_{x\in\partial\Gamma_T}=\varphi(x,t).$$

Here ($\varphi(x,t)$ is the boundary control of the system (2); $f(x,t)\in L_{2,1}(\Gamma_T)$, $\psi(x)\in L_2(\Gamma)$.

Definition 1. *A weak solution of the boundary value problem (2) is the function* $y(x,t)\in V^{1,0}(a,\Gamma_T)$, *that satisfies the integral identity*

$$\int_\Gamma y(x,t)\eta(x,t)dx-\int_{\Gamma_t} y(x.t)\eta_t(x,t)dxdt+ \\ +\ell_t(y,\eta)=\int_\Gamma \psi(x)\eta(x,0)dx+ \\ +\int_{\Gamma_t}\varphi(x,t)\eta(x,t)dxdt+\int_{\Gamma_t} f(x,t)\eta(x,t)dxdt$$

for any $t\in[0,T]$ *and any function* $\eta(x,t)\in W^1(a,\Gamma_T)$;

$\ell_t(y,\eta)$ *– bilinear form defined by the relation*

$$\ell_t(y,\eta)=\int_{\Gamma_t}(a(x)y_x(x,t)\eta_x(x,t)+ \\ +b(x)y(x,t)\eta(x,t))dxdt.$$

Theorem 1. P*roblem* (2) *is uniquely solvable in the space of weakly* $V^{1,0}(a,\Gamma_T)$.

The state $y(x,t)\in V_2^{1,0}(a,\Gamma_T)$ of the system (3) is determined by a weak solution $y(\nu)(x,t)$ of the problem (2) ($\varphi(x,t)=\nu(x,t)$); $L_2(\partial\Gamma_T)$ – the space observation: $Cy(\nu)=M\,y(\nu)\big|_{\partial\Gamma_T}$ ($M:L_2(\partial\Gamma_T)\to L_2(\partial\Gamma_T)$ – continuous linear operator, here $y(\nu)\big|_{\partial\Gamma_T}$ – trace of function $y(\nu)$ on the surface $\partial\Gamma_T$); functional $J(\nu)$ that requires minimization of a convex closed set $U\partial\subset U$ is of the form

$$J(\nu)=\mathrm{P}My(\nu)-z_0\mathrm{P}^2_{L_2(\partial\Gamma_T)}+(N\nu,\nu)_U$$

where $z_0(x,t)\in L_2(\partial\Gamma_T)$ – a predetermined observation.

For the problem (2) define the dual state $\omega(\nu)(x,t)\in W^1(a,\Gamma_T)$ as a function that satisfies the integral identity

$$-\int_{\Gamma_T}\omega(\nu)(x,t)\zeta(x,t)dxdt+\ell_T(\omega(\nu),\zeta)= \\ =\int_{\partial\Gamma_T}M^*(My(\nu)(x,t)-z_0(x,t))\zeta(x,t)dxdt$$

for all functions $\zeta(x,t)\in W^{1,0}(a,\Gamma_T)$.

Theorem 2. *To element* $u(x)\in U_\partial$ *was optimal control of the system* (2) *is necessary and sufficient that the following relationship are satisfy to:*

$$\int_\Gamma y(u)(x,t)\eta(x,t)dx-\int_{\Gamma_t} y(u)(x,t)\eta(x,t)dxdt+ \\ +\ell_t(y(u),\eta)=\int_\Gamma\psi(x)\eta(x,0)dx+ \\ +\int_{\partial\Gamma_t}u(x,t)\eta(x,t)dxdt+\int_{\Gamma_t}f(x,t)\eta(x,t)dxdt$$

for all $t\in[0,T]$ *and for all function* $\eta(x,t)\in W^1(a,\Gamma_T)$;

$$-\int_{\Gamma_T}\omega(u)(x,t)\zeta(x,t)dxdt+\ell_T(\omega(u),\zeta)= \\ =\int_{\partial\Gamma_T}M^*(My(u)(x,t)-z_0(x,t))\zeta(x,t)dxdt$$

for all functions $\zeta(x,t)\in W^{1,0}(a,\Gamma_T)$;

$$\int_{\partial\Gamma_T}(\omega(u)(x,t)+Nu(x,t))(\nu(x,t)-u(x,t))dxdt\ge 0$$

for all $\nu\in U_\partial$. *Here* $y(u)\in V^{1,0}(a,\Gamma_T)$, $\omega(u)\in W^1(a,\Gamma_T)$ *и* $\omega(u)(x,T)=0$, $x\in\Gamma$.

Solved the problem of synthesis of the optimal edge control for $U_\partial=U$.

2.2. Mathematical model with distributed parameters in the netlike domain

The ideas presented above apply to the multivariate case. Consider an open bounded domain Ψ of the Euclidean space R^n that has a netlike structure [3, 4], i.e., $\Psi=(\bigcup_k\Psi_k)\bigcup(\bigcup_l S_l)$, where S_l is a surface that separates adjacent domains Ψ_k, $\partial\Psi$ indicates the boundary of Ψ (initially, the smoothness of $\partial\Psi$ is not important). The locus of conjugation of the adjacent domains Ψ_k will be called the node locus and further denoted by ξ; it represents the union of surfaces $S_l(\xi)$ whose number coincides with the number of conjugated domains, that is, $\xi = \bigcup_l S_l(\xi)$. Next you enter the space $L_2(\Psi)^n$, $L_2(\Psi_T)^n$, and $V_0^{1,0}(\Psi_T)$ similar $L_2(\Gamma)$, $L_2(\Gamma_T)$ and $V^{1,0}(a,\Gamma_T)$; $V_0^{1,0}(\Psi_T)$ circuit elements $\phi \in D(\Psi)^n$ in norm $\|\phi\| = (\|\phi\|^2_{L_2(\Psi)} + \|\phi_x\|^2_{L_2(\Psi)})^{1/2}$, satisfies conditions

$$Y\big|_{S_l^-(\xi)} = Y\big|_{S_l^+(\xi)}, \quad \sum_l \frac{\partial Y}{\partial n_l^-}\bigg|_{S_l^-(\xi)} + \sum_l \frac{\partial Y}{\partial n_l^+}\bigg|_{S_l^+(\xi)} = 0$$

(here $D(\Psi_T)^n$ is infinitely differentiable in Ψ_T functions with compact supports in Ψ_T, $div\phi = 0$; $S_l^-(\xi)$ and $S_l^+(\xi)$ mean the unilateral surfaces for $S_l(\xi)$ defined by the direction of the normals n_l^- and n_l^+ to the surfaces $S_l^-(\xi)$ and $S_l^+(\xi)$, respectively).

For a vector function $Y(x,t) = \{y_1(x,t), y_2(x,t), \ldots, y_n(x,t)\} \in V_0^{1,0}(\Psi_T)$ ($x = \{x_1, x_2, ..., x_n\}$) defined in a domain $\Psi_T = \Psi \times (0,T)$ ($T < \infty$), consider the linearized Navier--Stokes system

(3)

$$Y_t - \nu\Delta Y + \nabla p = f, Y(x,0) = Y_0(x), x \in \Psi, Y\big|_{\partial\Psi} = 0$$

Theorem 3. *The initial boundary value problem* (3) *has a unique weak solution in the space* $V_0^{1,0}(a,\Psi_T)$. *A weak solution of the initial boundary value problem* (3) *continuously depends on the initial data* $f(x,t)$ *and* $Y_0(x)$ ***(****definition of weak problem solving* (3) *similar to definition* 1 *on* Γ_T *).*

Next, we study two types of optimal control problems that are common in applications, namely, distributed control and starting control (with terminal observations). In the former case, control action appears in the right-hand side of the Navier-Stokes system (i.e., defines the density of external forces); in the latter case, it defines the initial condition of the system at $t = 0$. In both cases, the physical problem is to speed up an incompressible viscous multiphase medium to a given vector velocity field by a given (terminal) time $t = T$.

3. Conclusions

Receive necessary and sufficient conditions for the existence of optimal control, similar to the representation in the theorem 2. The described algorithm is applicable to many optimization problems for differential systems whose states are defined by weak solutions of evolutionary equations on similar networks as in the papers [3 – 5]. Interestingly, other researchers considered alternative approaches to the stability analysis [6 – 8] and stabilization [9 – 11] of the solutions to some applications-relevant classes of complex systems, yet with the same treatment of the optimal control existence conditions.

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