FUNDAMENTALS OF THE KINETIC THEORY OF MULTICOMPONENT EMULSIONS

ОСНОВЫ КИНЕТИЧЕСКОЙ ТЕОРИИ МНОГОКОМПОНЕНТНЫХ ЭМУЛЬСИЙ

Prof., Dr. Tech. Sci. Firsov A.N.
Peter the Great St. Petersburg Polytechnic University – St. Petersburg, Russia
E-mail: anfirs@yandex.ru

Abstract: *The paper proposes a mathematical model for describing the dynamics of multicomponent emulsions, based on ideas and methods of the kinetic theory of gases. The methodological basis of the proposed theory is the ideas and methods of the theory of integral kinetic equations.*

Keywords: MULTICOMPONENT EMULSIONS, KINETIC THEORY, MATHEMATICAL MODEL, INTEGRAL KINETIC EQUATIONS

1. Introduction

In some practically important situations (such as the movement of oil through a well), one has to deal with the problem of the motion of a viscous fluid, inside which there are small inclusions in the form of gas bubbles, droplets of water, solid particles, etc. In this paper, we shall consider the case of emulsions[1]. This means that, on the one hand, the entire "mixture" (i.e., the liquid plus inclusions) can be considered as a continuous medium, and on the other hand, that in a small element of the volume of the medium there are "sufficiently many sufficiently small particles". In this connection, it seems quite natural to apply the statistical approach in the above-mentioned problem in the spirit of the kinetic theory of gases. However, the theoretical results known to the author in this direction [1 - 7] are connected with concrete (and rather simple) physical models, which does not allow us to sufficiently cover the problem of the motion of emulsions and give its closed mathematical formulation.

We see the main goal of the work, just in order to give a sufficiently general and precise mathematical formulation of this problem, which allows, in particular, to understand the place and role of some simplifying assumptions. From a methodological point of view, the author followed the basic ideas developed by the Leningrad school of aerodynamics of rarefied gas [8], founded by S. V. Vallander.

2. Statement of the problem of modeling multicomponent emulsions based on a kinetic approach

So, we will consider a viscous liquid, inside which there is a very large number of small "particles" (gas bubbles, drops of other liquids, etc.). The fact that these inclusions are emulsions means, in particular, that the whole mixture (liquid plus inclusions) - from a macroscopic point of view - can be considered as a continuous medium[2]. In this connection, it becomes necessary to distinguish three scales of smallness of distances (and volumes). The first of them corresponds to the concept of a small (elementary, physically infinitesimal) volume of a mixture in the hydrodynamic sense. This is the volume within which, on the one hand, the hydrodynamic and thermodynamic quantities related to the mixture as a continuous medium can be regarded as identical, and on the other hand, in which there are a sufficient number of inclusion particles, so that the latter can be applied a statistical approach.

The second scale corresponds to the concept of a small (from the hydrodynamic point of view) volume of the main fluid. This volume, generally speaking, is of a lower order than the previous one, and its linear dimensions are of the order of the average distance between the particles.

The third scale corresponds to the dimensions of the inclusion particles themselves. These dimensions will be assumed to be small of a higher order than the average distances between the particles.

As the basic elementary volume, it is natural to take an elementary volume corresponding to the first of the mentioned scales of smallness. Hydrodynamic and thermodynamic quantities characterizing the state of the main fluid will be understood as averaged over such an elementary volume.

Finally, we assume that the inclusion particles have a spherical shape and form a set for which one can use the assumptions commonly used in the definition of the concept of "rarefied gas" (the pairing of collisions, the negligibly small duration of the collision time in comparison with the time of free motion, etc., see, for example, [9]).

The nature of the interaction of particles with each other and with the main fluid requires special consideration. Here we will focus on those aspects of this interaction that are essential for the purposes of this article.

It is known that the gas bubbles differ significantly from the other particles (droplets of liquid, solid particles, etc.), both in terms of their individual properties and the effect on the dynamics of the mixture as a whole. First, this difference is manifested in the fact that the dimensions of gas bubbles can change during their free movement inside the main liquid. This change is due to a change in the temperature and pressure of the main fluid and, as a result, the temperature and pressure of the gas inside the bubbles. With a great degree of certainty, we can assume that at any time the gas inside the bubble is in thermodynamic equilibrium with the surrounding liquid (this means that the temperature and pressure of the gas and liquid are coincide). Thus, the radius r of the bubble is related to the temperature T and pressure p of the main liquid by formula

$$r = \left(\frac{3}{4} \frac{m}{\pi} \frac{RT}{\mu p} \right)^{\frac{1}{3}} \tag{1}$$

Where m - is the mass of gas in the bubble, μ - the molecular weight of the gas, R - the universal gas constant. In the general case, we can assume that r is a given function of m, T, p and μ.

The change in the size of the bubble also results the fact that the mixture (as a continuous medium) cannot be regarded as an incompressible fluid, even if the base liquid is incompressible.

The next feature of bubbles is the possibility of their emergence from "nothing" (for example, as a result of chemical reactions occurring in the main liquid), and as a result of their spontaneous decay. The latter, in the simplest case, can be

[1] The presence of gas bubbles essentially distinguishes the problem of the motion of emulsions from the problem of the motion of suspensions; the latter was quite well developed from different points of view [1 - 7].

[2] In order to distinguish the emulsion mixture as a continuous medium from the actual liquid, in which all these inclusions are found, we shall call the latter "basic liquid".

considered reliably occurring as a result of reaching a certain limiting bubble radius r_0.

Interaction of particles with each other is simply their direct collision. In this case, however, a phenomenon such as the fusion of two bubbles or droplets, as well as their crushing are possible. We assume that when two bubbles (or droplets) collide, either their merging is possible, or because of the collision, the number of particles does not change (that is, it remains equal to two), and in the collision of two particles of different types (a bubble with a droplet, etc.), these particles change only the velocities.

3. Basic properties of models and connection with problems of the theory of transfer

We now introduce the basic functions that we shall deal with below, and indicate some of their properties. We will mark the type of particle (bubble, drop, solid particle, etc.) by the indexes i, j, k, etc., taking integral values. The number of values of these indices is obviously finite.

For brevity, we introduce the following terminology. We say that some particle of the type i there is a particle of type (i, x, u, m), if this particle is reliably located at the point of space with a radius vector $x(x^1, x^2, x^3)$, has a speed $u(u^1, u^2, u^3)$ and mass m, and that this particle is of type (i, x, dx, u, du, m, dm), if its spatial coordinates are enclosed in the interval $[x^k, x^k + dx^k]$, $k = 1,2,3$, the projection of the velocities in the gaps $[u^k, u^k + du^k]$, $k = 1,2,3$, and the mass in the gap $[m, m + dm]$.

By the distribution function of the particles of the variety i we shall call the function $f_i(x, u, m, t)$, having the property that the quantity $f_i(x, u, m, t)dxdudm$ gives (up to small higher order) the mathematical expectation of the number of particles of type (i, x, dx, u, du, m, dm) at the moment t. As in [9], it is easy to see that $f_i dxdudm$ there is also a probability of detecting one particle in volume $dxdudm$.

Let us denote by $P_i^{(k)}(x, u, m, t)$ a function possessing the property that the quantity $P_i^{(k)}(x, u, m, t)dt$, $k = 2,3,...,$ is a probability of decay of a particle into k parts over a period of time from t to $t + dt$, if this particle at the time moment t was authentically of the type (i, x, u, m).

Let's denote by $\omega_i^k(u, m | u_1, m_1)$ the density of mathematical expectation (in the space of variables (u_1, m_1)) of the number of particles of the type $(i, u_1, du_1, m_1, dm_1)$, obtained as a result of authentically decay into k parts of a particle of the type (i, u, m).

Let us set

$$\hat{T}_i^k(x, u, m, t | u_1, m_1) \equiv P_i^{(k)}(x, u, m, t)\,\omega_i^k(u, m | u_1, m_1) \cdot \quad (2)$$

The following properties of functions ω_i^k и $\hat{T}_i^k$ are obvious:

$$\omega_i^k(u, m | u_1, m_1) = 0 \text{ при } m_1 \ge m \ , \quad (3)$$

$$\hat{T}_i^k(x, u, m, t | u_1, m_1) = 0 \text{ при } m_1 \ge m \ , \quad (4)$$

$$\int_0^\infty \omega_i^k(u, m | u_1, m_1)\, du_1 dm_1 = k, \quad (5)$$

$$\sum_{k=2}^\infty \frac{1}{k} \int \int_0^\infty \hat{T}_i^k(x, u, m, t | u_1, m_1)\, du_1 dm_1 = P_i\ (x, u, m, t) \quad (6)$$

Here

$$P_i\ (x, u, m, t) = \sum_{k=2}^\infty P_i^{(k)}(x, u, m, t)$$

gives, obviously, the probability of decay per unit time of a particle of the type (i, x, u, m) at the time t. Here and below, if the region of integration is not indicated, we mean the entire space R^3.

We note that, since the quantity $P_i^{(k)} / P_i$ is the probability of decay of a particle of the type (i, x, u, m) in k parts provided that some decay has occurred reliably, the value

$$\hat{T}_i\ (x, u, m, t | u_1, m_1) = \frac{1}{P_i} \sum_{k=2}^\infty P_i^{(k)}(x, u, m, t)\,\omega_i^k(u, m | u_1, m_1) \quad (7)$$

gives the density of mathematical expectation of the number of particles of type $(i, x, u_1, du_1, m_1, dm_1)$, in the space of variables (u_1, m_1), as a result of a reliable decay of a particle of the type (i, x, u, m).

By virtue of (3)

$$\hat{T}_i\ (x, u, m, t | u_1, m_1) = 0 \quad m_1 \ge m \cdot \quad (8)$$

We denote by $\hat{T}_{ij}^k(x, u_1, u_2, m_1, m_2 | u, m)$ the density of mathematical expectation (in the space of variables (u, m)) of a number of particles of type (k, x, u, du, m, dm), obtained as a result of a reliable collision of particles of types (i, x, u_1, m_1) and (j, x, u_2, m_2). We note that under the assumptions made in §2 $\hat{T}_{ij}^k = 0$ in the following cases:

1) $i = j, \quad k \ne i$;
2) $i \ne j, \quad k \ne i, j$;
3) $i \ne j, \quad k = i, \quad m \ne m_1;$
4) $i \ne j, \quad k = j, \quad m \ne m_2;$
5) $i = j = k, \quad m > m_1 + m_2.$

Thus, $\hat{T}_{ij}^k$ are nontrivial only for:

a) $i = j = k;$
b) $i \ne j, \quad k = i;$ c) $i \ne j, \quad k = j.$

In the cases b) and c) an expression for $\hat{T}_{ij}^k$ must contain factors of the form $\delta(m - m_1)$ or $\delta(m - m_2)$, so that in these cases

$$\int \int_0^\infty T_{iJ}^k(x, u_1, u_2, m_1, m_2 | u, m)\, du\ dm = 1 \quad (9)$$

Of particular interest is the case a), since here it is necessary to take into account the possibility of merging two particles of one kind. For brevity, we denote $T_{ii}^i \equiv T_i$. Let $h_i(x, u_1, u_2, m_1, m_2)$ is a probability of merging two reliably colliding particles of types (i, x, u_1, m_1) and (i, x, u_2, m_2), and $\tilde{P}_i(u_1, u_2, m_1, m_2 | u, m)$ is the probability density (in space (u, m)) of the fact that a particle, which as a result of a reliable fusion of these particles, will have the type (i,x,u,du,m,dm). We note at once that the expression for $\tilde{P}_i$ should contain a multiplier $\delta(m-(m_1+m_2))$. It's obvious that

$$\iint_0^\infty \tilde{P}_i \, du \, dm = 1 \tag{10}$$

Let's further $\tilde{T}_i(u_1,u_2,m_1,m_2|u,m)$ is the density of mathematical expectation (in the space of variables (u,m)) number of particles of type (i,x,u,du,m,dm), resulting from a reliable collision (without merging) of particles of the type (i,x,u_1,m_1) and (i,x,u_2,m_2). For $\tilde{T}_i$ (by the assumptions of §3) we have

$$\iint_0^\infty \tilde{T}_i \, du \, dm = 2 \tag{11}$$

In this way,

$$T_i(x,u_1,u_2,m_1,m_2|u,m) = h_i(x,u_1,u_2,m_1,m_2)\tilde{P}_i(u_1,u_2,m_1,m_2|u,m) = \\ =(1-h_i)\tilde{T}_i(u_1,u_2,m_1,m_2|u,m) \tag{12}$$

From (10) and (11) we obtain

$$\iint_0^\infty T_i \, du \, dm = 2-h_i \tag{13}$$

Suppose, finally, that the function $\Pi_i(x,u,m,t)$ is such that the quantity $\Pi_i dxdudmdt$ is a mathematical expectation of the number of particles of type (i,x,dx,u,du,m,dm), arising during a period of time from t to $t+dt$ as a result of processes not associated with collisions and particle decays (for example, as a result of chemical reactions in the main liquid).

We note that all the quantities introduced above can depend (as on the parameters) on the macroscopic characteristics of the main liquid at the corresponding point and at the corresponding instant of time.

In concluding this section, we introduce several functions that are important for the future, connected with the free motion of an individual particle of sort i.

Let at the moment t the particle under consideration is of the type (i,x,u,m). Then the equation of its motion in the main fluid (pressure, velocity, density and temperature are $p(x,t), v(x,t), \rho(x,t), T(x,t)$ respectively) in the presence of acceleration of gravity g, will have the form

$$\dot{x}(\tau) = u(\tau),$$

$$\dot{u}(\tau) = g - \frac{1}{m}\rho V_i(m,p_\tau,T_\tau) + \frac{1}{m}F_{conp}^{(i)}(m,V_i,p_\tau,T_\tau,v_\tau,u(\tau)) \equiv \\ \equiv G_i(m,p_\tau,T_\tau,v_\tau,u(\tau)). \tag{14}$$

These equations must be solved under conditions

$$x(\tau)\big|_{\tau=t} = x, \qquad u(\tau)\big|_{\tau=t} = u. \tag{15}$$

Here we introduce the following notation:

$$\dot{u} \equiv \frac{du}{d\tau}; \quad p_\tau = p(x(\tau),\tau); \quad T_\tau = T(x(\tau),\tau); \quad v_\tau = v(x(\tau),\tau);$$

V_i - denotes the volume of the particle under consideration, $F_{conp}^{(i)}$ - denotes the force of resistance to movement of a particle in the main fluid. In the simplest case, for $F_{conp}^{(i)}$ one can use the Stokes formula

$$F_{conp}^{(i)} = 6\pi\eta r a_i(v-u), \tag{16}$$

Where η - denotes dynamic viscosity of the main fluid; r - particle radius; $a_i = 1$, if it is a solid particle, and $a_i = \frac{1}{3}\frac{2\eta+3\eta_i'}{\eta+\eta_i'}$, if it is a drop or a gas bubble (here η_i' - dynamic viscosity of a liquid (gas) forming a droplet (bubble)).

Further, we denote by

$$x_\tau^{(i)} \equiv \varphi_i(\tau; x, u, m, t), \quad u_\tau^{(i)} \equiv \psi(\tau; x, u, m, t) \tag{17}$$

the solution of problem (14) - (15). Clearly,

$$x_\tau^{(i)} = x, \quad u_\tau^{(i)} = u \tag{18}$$

For each fixed τ functions (18) give a diffeomorphism of the phase space (x,u) into itself. We denote by $D_\tau^{(i)} = D_\tau^{(i)}(x,u,m,t)$ Jacobian

$$D_\tau^{(i)} = \frac{D(\varphi_i,\psi_i)}{D(x,u)} \tag{19}$$

In particular, an element of the volume of the phase space $dx_\tau^{(i)}du_\tau^{(i)}$ is associated with an element $dxdu$ by the relation

$$dx_\tau^{(i)}du_\tau^{(i)} = \left|D_\tau^{(i)}\right| dxdu \tag{20}$$

We also note that $D_t^{(i)} = 1$.

4. *Derivation of integral kinetic equations of the theory of multicomponent emulsions*

The principle of the derivation of a system of integral kinetic equations for functions f_i basically does not differ from usual [9, 10]. We, therefore, indicate here only the necessary changes.

4.1. *Probability of free movement.*

Let $Q_i(x,u,m,t)$ is the probability of collision or decay per unit time of the particle, which at the moment t authentically had the type (i,x,u,m).

For Q_i we have the expression

$$Q_i(x,u,m,t) = \sum_j \iint_0^\infty f_j(x,u_3,m_3,t)\left(r^{(i)}+r_3^{(j)}\right)|u_3-u|du_3dm_3 + \\ +P_i(x,u,m,t), \tag{21}$$

where $r^{(i)}, r_3^{(j)}$ - denote radii of particles (i,x,u,m) and (j,x,u_3,m_3) respectively. We emphasize that $r^{(i)}$ и $r_3^{(j)}$ depend, generally speaking, on $x,t,m(m_3)$ - see the formula (1).

Let's further $R_i(x,u,m,t,t+s)$ is a probability of a free (without collision and decay) motion during the time from t to $t+s$ of a particle, which at the moment t authentically had the type (i,x,u,m). Then

$$R_i(x,u,m,t,t+s) = \exp\left[-\int_t^{t+s} Q_i\left(x_\sigma^{(i)},u_\sigma^{(i)},m,\sigma\right)d\sigma\right] \tag{22}$$

We note that when the expression (21) is substituted into (22), $r^{(i)}, r_3^{(j)}$ should be replaced by $r_\sigma^{(i)}, r_{3,\sigma}^{(j)}$.

4.2. *Birth function.*

We denote by $\phi_i(x,u,m,t)$ the birth function of particles of sort i, that is, a function possessing the property that the quantity

$$dn_0^{(i)} = \phi_i(x, u, m, t)\,dxdudmdt$$

is a mathematical expectation of the number of particles of type (i, x, dx, u, du, m, dm), born within a period of time from t to $t+s$. Clearly,

$$dn_0^{(i)} = dn_1^{(i)} + dn_2^{(i)} + dn_3^{(i)},$$

where $dn_1^{(i)}$ is a mathematical expectation of the number of particles of this type, born as a result of collisions, $dn_2^{(i)}$ - born as a result of decay, a $dn_3^{(i)}$ - arising from other causes.

Using the notation of the preceding section, we obtain

$$dn_2^{(i)} = = dxdudmdt\int_0^\infty\int f_i(x, u_1, m_1, t)P_i(x, u_1, m_1, t)\hat{T}_i\,(x, u_1, m_1, t|u, m)\,du_1 dm_1;$$

$$dn_3^{(i)} = \Pi_i(x, u, m, t)\,dxdudmdt.$$

The expression for $dn_1^{(i)}$ has the usual form [10]. In this way,

$$\phi_i(x, u, m, t) = \frac{\pi}{2}\sum_{j,k}\int_0^\infty\int_0^\infty\int\int |u_1 - u_2|\left(r_1^{(j)} + r_2^{(k)}\right)^2 f_j(x, u_1, m_1, t)\, f_k(x, u_2, m_2, t)\times$$
$$\times T_{jk}^i(x, u_1, u_2, m_1, m_2|u, m)\,du_1du_2dm_1dm_2 + \int_0^\infty\int f_i(x, u_1, m_1, t)P_i(x, u_1, m_1, t)\times$$
$$\times\hat{T}_i\,(x, u_1, m_1, t|u, m)\,du_1dm_1 + \Pi_i(x, u, m, t) \tag{23}$$

The system of integral kinetic equations for f_i is output in the standard way [9], and has the form (in the absence of boundaries[3])

$$f_i(x, u, m, t) = f_i\left(x_{t_0}^{(i)}, u_{t_0}^{(i)}, m, t_0\right)\times\exp\left[-\int_{t_0}^t Q_i\left(x_q^{(i)}, u_q^{(i)}, m, q\right)dq\right]\times$$
$$\times\left|D_{t_0}^{(i)}\right| + \int_{t_0}^t \phi_i\left(x_\tau^{(i)}, u_\tau^{(i)}, m, \tau\right)\times\exp\left[-\int_\tau^t Q_i\left(x_q^{(i)}, u_q^{(i)}, m, q\right)dq\right]\left|D_\tau^{(i)}\right|d\tau. \tag{24}$$

Here $t_0 < t$ - is arbitrary time moment.

A simple technique analogous to that used in [11] and consisting in the passage to the limit $t \to t_0$, enables us to obtain from (24) the following system of integro-differential equations

$$\frac{\partial f_i}{\partial t} + u\cdot\frac{\partial f_i}{\partial x} + \frac{\partial (G_i f_i)}{\partial u} = \phi_i - f_i Q_i, \tag{25}$$

Where G_i is given by the relation (14) and all functions are taken at the point (x, u, m, t).

5. *Analysis of the relationship between the macroparameters of the mixture and the main liquid*

If given $p(x,t), v(x,t),\ T(x,t)$ then equations (24) together with equalities (21), (23) form a closed system. However, in practice it is difficult to determine the quantities characterizing the main liquid. On the other hand, (in particular, experimentally) values $p_c(x,t), v_c(x,t),\ T_c(x,t), \rho_c(x,t)$, characterizing the pressure, velocity, temperature, and density of the mixture as a continuous medium, can be found. Therefore, in order to close the system of equations, it is necessary to add the relations connecting $f_i,\ p_c,\ v_c,\ T_c, \rho_c, p\ ,\ v\ ,\ T\ , \rho$. Let's do it. In the future, it will be more convenient to use the value of the internal energy density instead of the temperature. So, let $E(x,t)$ - is a density (per unit mass) of the internal energy of the main fluid. $E_c(x,t)$ - density (per unit mass) of the internal energy of the mixture (as a continuous medium) and e_i - density (per unit mass) of the internal energy of the substance (gas, liquid) in the particle of the variety i. With the assumptions made in §3, we can assume that $e_i = e_i(p, T)$. Let dx - a certain elementary volume of the mixture (see §3) adjacent to the point x. Obviously,

$$dx = d_1x + d_2x,$$

where d_1x - is a part of the volume dx, occupied by particles, and d_2x - part of the volume dx, occupied by basic fluid.

We have

$$d_1x = dx\sum_j\int_0^\infty\int V_i(m, p\ , v\ , T)\, f_i(x, u, m, t)\,dudm$$

where V_i - is a volume of a particle of a variety i.

The mass of the particles inside dx will be equal

$$d_1M = dx\sum_i\rho_i\int_0^\infty\int V_i f_i\ dudm = dx\sum_i\int_0^\infty\int m\ f_i\ dudm,$$

where ρ_i - is a density of matter in the corresponding particle.

The mass of the main liquid inside dx is equal, hence,

$$d_2M = dx\rho\left(1 - \sum_i\int_0^\infty\int V_i f_i\ dudm\right).$$

Thus, for the density of the mixture we obtain

$$p_c(x,t) = \rho - \sum_i\left(\frac{\rho}{\rho_i} - 1\right)\int_0^\infty\int m\ f_i\ dudm \tag{26}$$

Let us turn to the expression for the macroscopic velocity of the mixture $v_c(x,t)$. We use the formula

$$d\mathrm{K}_c = v_c\,\rho_c\,dx,$$

where $d\mathrm{K}_c$ - is the amount of motion of the mixture inside dx. Taking into account that $d\mathrm{K}_c$ is composed of the amount of motion of the particles and the main fluid, we obtain

$$v_c = \frac{1}{\rho_c}\left[\rho v\left(1 - \sum_i\frac{1}{\rho_i}\int_0^\infty\int m\ f_i\ dudm\right) + \sum_i\int_0^\infty\int mu\ f_i\ dudm\right] \tag{27}$$

To find the expression for E_c, note that the total energy of the mixture inside dx is made up of the kinetic and internal energies of the particles and the main fluid. Carrying out the corresponding calculations, we obtain

$$E_c = U_c / \rho_c - v_c^2 / 2, \tag{28}$$

Where U_c - is the bulk density of the total energy of the mixture:

$$U_c = E\ \rho - \sum_i\left(\frac{E\rho}{e_i\rho_i} - 1\right)\int_0^\infty\int m\, e_i\, f_i\ \ dudm + \tag{29}$$
$$+\frac{\rho v^2}{2}\left(1 - \sum_i\frac{1}{\rho_i}\int_0^\infty\int m\ f_i\, dudm\right) + \sum_i\int_0^\infty\int\frac{mu^2}{2}\, f_i\, dudm$$

Formulas (26) - (29) give the required additional relations for the closure of the system of equations (24).

6. *Concluding remarks*

Let us now make a few general remarks.

1. Equations (24), (25) are externally similar to the known kinetic equations for gas mixtures [10], but their content is largely different. In particular, under the assumption that there are no collisions between the particles in the problem under consideration,

[3] In practically interesting problems, we can assume that there are no solid boundaries in the mixture, or on them the functions f_i are given.

the right-hand side of (25) does not vanish:

$$\phi_i - f_i Q_i = \int\int_0^\infty f_i\left(x, u_1, m_1, t\right) P_i\left(x, u_1, m_1, t\right) \times$$
$$\times \hat{T}_i\left(x, u_1, m_1, t \middle| u, m\right) du_1 dm_1 + \Pi_i\left(x, u, m, t\right) - \tag{30}$$
$$- f_i\left(x, u, m, t\right) P_i\left(x, u, m, t\right)$$

2. «Equilibrium solution» (that is, a solution that does not depend on x, t) plays a somewhat different role in the problem under study than in the kinetic theory of gases. This is because the dependence on the coordinate is laid inside the very essence of the problem: macro parameters $p(x,t)$, $T(x,t)$ etc. *a priori* are changing over *x* very quickly.

The latter means that solutions close to equilibrium are of little interest from a practical point of view, and therefore the equilibrium solutions themselves are of little interest. They, however, can be considered as limiting (at $|x| \to \infty$) function values f_i; so the equilibrium solutions can be used mainly in the formulation of boundary value problems in unbounded domains.

On the other hand, stationary solutions may be the most interesting from a practical point of view.

3. In the problem of the motion of emulsions, the viscosity of the main liquid plays an important role, and, generally speaking, it can not be neglected. Indeed, if we neglect the term $F_{conp}^{(i)}$ in (14), then any freely moving particle, after a sufficiently long time, will have an arbitrarily high velocity. Consequently, it may turn out that

$$|u|, |x|, t \to \infty \quad f_i \not\to 0,$$

but it is unreasonable.

4. In general case, theoretical search of functions P_i, $\hat{T}_i, T_i^k$ etc., defined in §3, is a very difficult task and is currently hardly feasible. In this connection, in our opinion, the role of experimental studies in this direction increases.

We indicate, however, the formula for $P_i\left(x, u, m, t\right)$ in the event that the decay of a particle of a variety *i* occurs if and only if its radius reaches a certain limiting value $r_0^{(i)}$. As already mentioned, we can assume that $r = \theta_i\left(p(x,t), T(x,t), m\right)$.

Let

$$m_0^{(i)} = m_0^{(i)}\left(p, T, r_0^{(i)}\right)$$

is such that

$$r_0^{(i)} = \theta\left(p, T, m_0^{(i)}\right).$$

Then

$$P_i\left(x, u, m, t\right) = u \cdot \left(\frac{\partial \theta_i}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial \theta_i}{\partial T}\frac{\partial T}{\partial x}\right)\left|\frac{\partial \theta_i}{\partial m}\right|^{-1} \delta\left(m - m_0^{(i)}\right)$$

5. In the previous sections we did not take into account the possibility of the proper rotation of the inclusion particles. Accounting for this possibility does not cause fundamental difficulties in the sense of deriving the basic equations, but the solution (in particular, numerical) of these equations is significantly complicated because of the increase in the number of independent variables for functions f_i. On the other hand, in the broad class of practically interesting cases, the assumption of the absence of proper rotation of spherical particles is justified [1, 2].

7. *References*

1. Fortier A. Mécanique des suspensions. Masson et C-ie Ėditeurs, Paris-VI[e], 1967.

2. Reologiya suspenziy. Seriya: Biblioteka Sbornika «Mekhanika»: Sbornik statey. M.: Mir, 1975. [Rheology of suspensions. Series: Library Collection "Mechanics"]. Moscow: Mir, 1975. (In Russian)

3. Bibik Ye.Ye. Reologiya dispersnykh sistem. L.: Izd-vo Leningr. un-ta, 1981. [Bibik E.E. Rheology of disperse systems]. L.: Leningrad University, 1981. (In Russian)

4. Kelbaliyev G. I. i dr. Mekhanika i reologiya neftyanykh dispersnykh sistem. M.: Izd-vo «Maska», 2017. [Kelbaliev G. I. et al., Mechanics and rheology of petroleum disperse systems]. Moscow: "Mask", 2017. (In Russian)

5. Tsibarov V.A. Kineticheskaya model' psevdoozhizhennogo sloya. – *Vestn. Leningr. un-ta*, 1975, № 13, s. 106-111. [Tsibarov V.A. Kinetic model of a fluidized layer]. – Vestnik of Leningrad University, 1975, No. 13, p. 106 - 111. (In Russian)

6. Pokrovskiy V.N. Statisticheskaya mekhanika razbavlennykh suspenziy. M.: Izd-vo «Nauka», 1978. [Pokrovsky V.N. Statistical mechanics of dilute suspensions]. Moscow: Nauka, 1978. (In Russian)

7. Protod'yakonov I.O., Tsibarov V.A., Chesnokov YU. G. Kineticheskaya teoriya gazovzvesey. L.: Izd-vo Leningr. un-ta, 1985. [Protodyakonov I.O., Tsibarov V.A., Chesnokov Yu.G. Kinetic theory of gas-weights]. L.: Leningrad University, 1985. (In Russian)

8. *Aerodinamika razrezhennykh gazov*. Vyp. 1 – 11. Sbornik statey. Pod red. S.V. Vallandera, R.G. Barantseva, L.: Izd-vo Leningr. un-ta, 1963 – 1983. [Vallander S.V., Barantsev R.G. (editors), Aerodynamics of rarefied gases]. Issues 1 – 11, L.: Leningrad University, 1963 - 1983. (In Russian)

9. Vallander S.V. Uravneniya i postanovka zadach v aerodinamike razrezhennogo gaza. – *Aerodinamika razrezhennykh gazov*. Vyp. 1. L.: Izd-vo Leningr. un-ta, 1963, s. 7-37. [Vallander S.V.. Equations and statement of problems in aerodynamics of a rarefied gas]. - In: Aerodynamics of rarefied gases. Issue. 1. L.: Leningrad University, 1963, p. 7-37. (In Russian)

10. Vallander S.V., Belova A.V. Integral'nyye kineticheskiye uravneniya dlya smesi gazov s vnutrennimi stepenyami svobody. - V kn.: *Aerodinamika razrezhennykh gazov*. Vyp. 1. L.: izd-vo Leningr. un-ta, 1963, s. 45-52. [Vallander S.V., Belova A.V. Integral kinetic equations for a mixture of gases with internal degrees of freedom]. - In: Aerodynamics of rarefied gases. Issue 1. L.: Leningrad University, 1963, p. 45-52. (In Russian)

11. Filippov B.V. Variant nestatsionarnykh kineticheskikh uravneniy. - V kn.: *Aerodinamika razrezhennykh gazov*. Vyp. 1. L.: izd-vo Leningr. un-ta, 1963, s. 67-73. [Filippov B.V. Variant of non-stationary kinetic equations]. - In: Aerodynamics of rarefied gases. Issue 1. L.: Leningrad University, 1963, p. 67-73. (In Russian)