PARAMETRIC INDUCED INSTABILITIES OF BOSONS IN MAGNETAR'S CRUST

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Abstract: *In the present work, we are discussing the Klein--Gordon equation describing relativistic spinless particles evolving in the (stationary) magnetar's crust. With the wave function expressed in terms of Mathieu's functions, we compute first-order transition amplitudes, pointing out the role of the strong magnetic induction in transitions to states which may be characterized by values of the model's parameters in instability bands.*

***KEYWORDS**: KLEIN-GORDON EQUATION, MATHIEU'S EQUATION, BESSEL FUNCTIONS, PERTURBATION THEORY, MAGNETARS*

1. Introduction

In 1992, Duncan and Thompson introduced the term of magnetar for a type of neutron star with massively boosted magnetic fields whose decay powers the emission of high-energy electromagnetic radiation [1]. The configuration of the background field inside has been extensively worked out and is not unique. Inspired by some previous investigations [2-4], we are assuming a radial magnetic induction, parallel to the surface of a plane slab with finite thickness in the *z*-direction.

As for the matter content, in the magnetar's crust and core, one may expect the existence of various particles, including boson condensates, which might have a significant influence on the star properties. In this respect, the kaons are seen as best candidates, besides nucleons and leptons [5, 6].

In the present work, we are extending our previous results on boson's wave function, solution to the Klein-Gordon equation [3, 4]. Thus, within a perturbative approach, we are discussing special cases for parameters' ranges which are leading to non-trivial quantization laws and to a favored distribution of the magnetic field inside the crust, corresponding to a high probability of transitions.

With reference to the Mathieu's equation, challenging aspects that surround the interest around it and its solutions are more and more invoked as a solid motivation as these manifest in a rich manner, both theoretically – mostly, from algebraic point of view – and practically, i.e. from the applicative side. Concerning the latter, the range of physical situations in which Mathieu functions appear is extremely extended, from elastic wave equations worked in elliptical boundary geometries as it is the case of resonators or waveguides, the motion of quantum particles in a periodic type potential or the stability of floating vessels for harmonious coherent waves of various frequencies and swings [7], to some modern applications as the physics of a capacitor microphone, ferromagnetic substance manifesting elastic oscillations [8], the particle's behavior in different systems of electromagnetic traps [9] or the quantum dynamics of the electrons within the free electron lasers (FEL) [10].

From algebraic point of view, manipulating Mathieu functions is not at all an easy task. The algebra behind the Mathieu functions is extremely intricate, still admitting a range of unsolved mathematical and computational issues. For instance, if we take into account the convergence of Mathieu functions, some authors report slow convergence or the lack of convergence of specific representations of Mathieu functions [11]. Some codes [12], which led to results with single-precision accuracy (7 decimal places), proved to respond negatively to the action of extending them to get a double-precision accuracy (15 decimal places).

An ardent area of research on Mathieu functions focuses on deriving expressions for the dependence of the *Characteristic Exponent* γ on the Mathieu equation parameters, α, β. Due to the physical significance of the Characteristic Exponent, which proves to be intimately connected to the wavelike nature of the physical phenomena modelled by the Mathieu equation, finding relations of the form $\gamma = \gamma(\alpha, \beta)$ constitutes an outstanding mathematical approach. In literature, relations of this type written in power series of β, can be found both for integer [13, 14] and noninteger values of γ [15].

We point out that the literature is pretty rich in treating specific algebraic aspects of Mathieu functions and the main difficulties within the computational analysis [16, 17, 18, 19, 20].

To put it concisely, developing packages for calculating Mathieu functions and operating with them represents an instigation for the modern research being an open territory for algebraic and computational explorations.

2. Magnetic field configuration in the crust

Let us consider, besides the vertical induction B_z, a radial component parallel to the surface of the slab and vanishing at $z = 0$ and $z = L$, of the form $B_\rho = b(t)\sin(\kappa z)$ [3, 4]. Working in cylindrical coordinates, the potential component

$$A_\varphi = \frac{B_0}{\kappa} exp\left[-\frac{\kappa^2}{\sigma} t\right] cos(\kappa z) \quad (1)$$

is generating the following electromagnetic field configuration

$$E_\varphi = -\frac{\partial A_\varphi}{\partial t} = \frac{\kappa B_0}{\sigma} exp\left[-\frac{\kappa^2}{\sigma} t\right] cos(\kappa z)$$
$$B_\rho = -\frac{\partial A_\varphi}{\partial z} = B_0 exp\left[-\frac{\kappa^2}{\sigma} t\right] sin(\kappa z)$$
$$B_z = \frac{1}{\rho} A_\varphi = \frac{B_0}{\kappa\rho} exp\left[-\frac{\kappa^2}{\sigma} t\right] cos(\kappa z) \quad (2)$$

which also satisfy the Maxwell equations.

In the following, we are assuming that the time variable is much less the *characteristic time*, $t \ll \tau = \sigma/\kappa^2 \approx 1 Myr$, which is comparable to the average Ohmic timescale, so that the exponential function in (2) can be set to one. However, soon after the crust forms, the magnetic field is freezing and such objects can be treated as being stationary, with poloidal and toroidal fields of the same order of magnitude [21, 22].

In order to study the semi-classical behavior of the charged scalar field, we start with the Klein-Gordon equation

(3) $$\left\lfloor \eta^{ij} D_i D_j - m_0^2 \right\rfloor \Phi = 0,$$

where the gauge derivatives are expressed in terms of the four-potential components as $D_i = \partial_i - iqA_i$.

For the essential component

(4) $$A_\varphi = \frac{B_0}{\kappa} cos(\kappa z),$$

the Klein-Gordon equation, in cylindrical coordinates,

$$\left[\Delta - \partial_t^2 - m_0^2 \right] \Phi = \frac{2iq}{\rho} A_\varphi \frac{\partial \Phi}{\partial \varphi} + q^2 A_\varphi^2 \Phi,$$

with the variables separation

(5) $$\Phi = \phi(\rho, z) e^{im\varphi} e^{-i\omega t},$$

leads to the following differential equation for $\phi(\rho, z)$,

(6) $$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \phi}{\partial \rho}\right) + \frac{\partial^2 \phi}{\partial z^2} + \left[\omega^2 - m_0^2 - \frac{m^2}{\rho^2} - \left(\frac{qB_0}{\kappa}\right)^2 cos^2(\kappa z)\right]\phi = -\frac{2mqB_0}{\kappa\rho} cos(\kappa z)\phi.$$

By identifying the potential operator

(7) $$\hat{V} = -\frac{2mqB_0}{\kappa \rho} cos(\kappa z),$$

the equation (6) gets the standard form employed in the Perturbation Theory, $\hat{\mathbf{D}}\phi = V\phi$, where $\phi = \psi + \chi$.

The zero-order equation

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi}{\partial \rho}\right) + \frac{\partial^2 \psi}{\partial z^2} + \left[\omega^2 - m_0^2 - \frac{m^2}{\rho^2} - \left(\frac{qB_0}{\kappa}\right)^2 cos^2(\kappa z)\right]\psi = 0$$

for $\psi(\rho, z) = F(\rho)T(z)$, splits into the following system

(8) $$\frac{d^2F}{d\rho^2} + \frac{1}{\rho}\frac{dF}{d\rho} + \left[\omega^2 - m_0^2 - p_z^2 - \frac{m^2}{\rho^2}\right]F = 0, \quad (a)$$

$$\frac{d^2T}{dz^2} + \left[p_z^2 - \frac{1}{2}\left(\frac{qB_0}{\kappa}\right)^2 - \frac{1}{2}\left(\frac{qB_0}{\kappa}\right)^2 cos(2\kappa z)\right]T = 0. \quad (b)$$

The first equation in (8) is satisfied by the Bessel functions

(9) $$F_m(\rho) = J_m(P\rho), \text{ with } P \equiv \sqrt{\omega^2 - m_0^2 - p_z^2},$$

while the second one can be identified with the Mathieu-type equation [23]

(10) $$y'' + \left\lfloor \alpha - 2\beta cos(2\zeta) \right\rfloor y = 0,$$

of parameters

(11) $$\alpha = \frac{p_z^2}{\kappa^2} - \frac{1}{2}\left(\frac{qB_0}{\kappa^2}\right)^2, \quad \beta = \frac{1}{4}\left(\frac{qB_0}{\kappa^2}\right)^2,$$

and variable $\zeta = \kappa z$. The general solutions of the equation (10),

(12) $$T(\alpha, \beta, \zeta) = \left\{MathieuC[\alpha, \beta, \zeta], MathieuS[\alpha, \beta, \zeta]\right\}$$

have been discussed in detail in our previous papers [3, 4], especially with respect to their stability. In general, the Mathieu's functions are of the form $f(z) \sim e^{i\gamma z} u(z)$, where the Mathieu Characteristic Exponent (MCE), γ, may be real or imaginary, depending on the values of the model parameters.

Nevertheless, we find worth stressing some interesting mathematical properties of these solutions. Thus, the even and odd solutions, can be written as the Fourier series [23]

(13.a) $$MathieuC = ce_{2n} = \sum_{j=0}^{\infty} A_{2j}^{2n}(\beta) cos(2j\zeta)$$

(13.b) $$MathieuS = se_{2n+1} = \sum_{j=1}^{\infty} B_{2j+1}^{2n+1}(\beta) sin((2j+1)\zeta)$$

In explicit calculations, one may employ the periodic expressions of the Mathieu's functions (13.a), valid in the first order in β

$$\begin{aligned} ce_0(z, \beta) &= \sum_{j=0}^{\infty} A_{2j}^0(\beta) cos(2jz) = A_0^0 + \sum_{j=1}^{\infty} A_{2j}^0(\beta) cos(2jz) \\ &= \frac{1}{\sqrt{2}}\left[-1 + J_0\left(\sqrt{\beta} exp(-iz)\right) + J_0\left(\sqrt{\beta} exp(iz)\right)\right] \\ &\approx \frac{1}{\sqrt{2}}\left[1 - \frac{\beta}{2} cos(2z) + \ldots\right] \end{aligned}$$

$$\begin{aligned} ce_2(z, \beta) &= \sum_{j=0}^{\infty} A_{2j}^2 cos(2jz) = A_0^2 + A_2^2 cos(2z) + \\ &+ \sum_{j=1}^{\infty} A_{2(j+1)}^2(\beta) cos(2(j+1)z) \\ &= \frac{\beta}{4} + \frac{4}{\beta}\left[J_2\left(\sqrt{\beta} exp(-iz)\right) + J_2\left(\sqrt{\beta} exp(iz)\right)\right] \\ &\approx cos(2z) + \frac{\beta}{12}(3 - cos(4z)) + \ldots \end{aligned}$$

where $J_n\left(\sqrt{\beta} exp(\pm iz)\right)$ are the Bessel functions and we have used the formulas [24]

$$A_{2j}^0 \approx \frac{2(-1)^j}{(j!)^2}\left(\frac{\beta}{4}\right)^j A_0^0 \; ; \; A_{2(j+1)}^2 \approx \frac{2(-1)^j}{j!(j+2)!}\left(\frac{\beta}{4}\right)^j;$$

$$A_0^0 = \frac{1}{\sqrt{2}} \; ; \; A_0^2 = \frac{\beta}{4}.$$

3. *First-order transition amplitude*

With the wave function

(14) $$\psi_m = J_m(P\rho) T(\alpha, \beta, \zeta) e^{im\varphi} e^{-i\omega t},$$

one may compute the one may compute the azimuthal current density component, of quantum origin, which is sustaining the magnetization current,

$$\begin{aligned} j_\varphi &= -\frac{iq}{\rho}\left(\psi_m^* \partial_\varphi \psi_m - \psi_m \partial_\varphi \psi_m^*\right) - 2q^2 |\psi_m|^2 A_\varphi \\ &= 2qJ_m^2 |T|^2 \left[\frac{m}{\rho} - \frac{qB_0}{\kappa} cos(\kappa z)\right]. \end{aligned}$$

In the transition amplitude

(15) $$\mathcal{A} = \int \psi^*_{m'} V \psi_m \rho d\rho d\varphi dz dt = -\frac{4\pi m b}{\kappa} I_1 I_2$$

where $b \equiv qB_0$, the first integral can be written in terms of the hypergeometric function F_{12} as [23]

(16) $$I_1 = \int_0^\infty J_m(P'\rho) J_m(P\rho) d\rho = \frac{1}{P'}\left(\frac{P}{P'}\right)^m \frac{\Gamma(m+1/2)}{\sqrt{\pi}\,\Gamma(m+1)} F_{12}\left[\frac{1}{2}, m+\frac{1}{2}, m+1, \left(\frac{P}{P'}\right)^2\right],$$

with $P' > P$. Once we fix the quantum number m, the integral above is expressed as a combination of *EllipticE* and *EllipticK* functions, of variable $(P/P')^2$.

The second integral in (15), i.e.

(17) $$I_2 = \int T^*(\alpha', \beta, \kappa z) cos(\kappa z) T(\alpha, \beta, \kappa z) dz,$$

should be analyzed for specific ranges of the particle momentum along *Oz*.

For small values of the parameter β, the theory of the Mathieu's functions is well understood. The characteristic values $\alpha_n(\beta)$ are important since they yield periodic solutions and they separate the regions of stability. The resonance condition $\alpha_n \approx n^2$ leads to the momentum quantization relation

(18) $$p_z^2 \approx n^2\kappa^2 + \frac{b^2}{2\kappa^2},$$

and to a quantized variable in the Bessel functions (9), i.e.

(19) $$P_n^2 = \omega^2 - m_0^2 - \frac{b^2}{2\kappa^2} - n^2\kappa^2.$$

For the particle momentum along *Oz* in the first band of instability

(20) $$\kappa^2 - \frac{b^2}{4\kappa^2} < p_z^2 < \kappa^2 + \frac{3b^2}{4\kappa^2},$$

the imaginary part of the MCE comes into play, $Im(\gamma) \sim \beta/2$, being responsible for the exponentially growing of the oscillatory wave functions.

By inspecting the stability chart of the Mathieu's functions, one can notice that, for increasing β, the stable regions situated between the characteristic curves $\alpha_n(\beta)$ become more and more narrow and their width is decreasing exponentially to practically discrete eigenvalues. For $\beta > 1$, one has a broad resonance and the solutions (12) are oscillating (along *Oz*) much faster than the electromagnetic field components, the ratio between the frequencies being $\sqrt{\beta} = b/(2\kappa^2)$.

In the case of a magnetar, whose magnetic field is extremely strong, $B_0 \sim 10^{10} - 10^{12}\,(T)$, the parameter β, proportional to the square of the magnetic induction, gets very large values. Thus, for $0 < \alpha < \beta$, i.e.

(21) $$\frac{b^2}{2\kappa^2} < p_z^2 < \frac{3}{4}\frac{b^2}{\kappa^2},$$

one has to use the following asymptotic expansion for the characteristic values [25]

(22) $$\alpha_n \approx -2\beta + 2(2n+1)\sqrt{\beta}.$$

In our case, the above relation turns into

(23) $$\alpha_n = \frac{b}{\kappa^2}\left\lfloor (2n+1) - \frac{b}{2\kappa^2} \right\rfloor,$$

leading to the Landau-type quantization law for the particle momentum along *Oz*

(24) $$p_z^2 \approx (2n+1)b,$$

and to a quantized variable in the Bessel functions computed with

(25) $$P_n^2 = \omega^2 - m_0^2 - (2n+1)b.$$

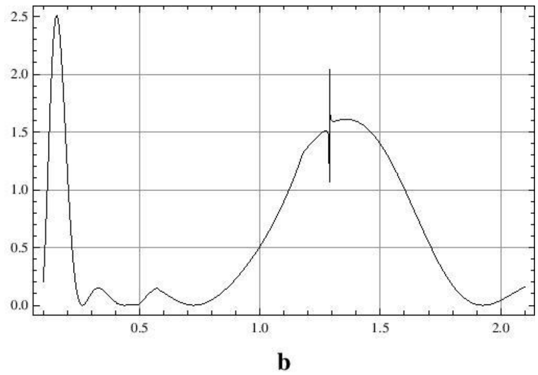

fig. 1 The absolute value of (17) as a function of $\mathrm{b} = b/\kappa^2$

In the figure 1, for the transition between $n = 2$ and $n' = n+1$, we are giving, in dimensionless units, the graphical representation of the absolute value of (17), as a function of the parameter $\mathrm{b} = b/\kappa^2\ b$, characterizing the strength of the magnetic field inside the crust, for a given κ. For both α_n and α_{n+1} positive quantities computed with (23), one may notice the peaks separated by zero minima, corresponding to high probabilities and zero probabilities, respectively.

Let us point out that, for an electron with the momentum along *Oz* given by (24), the distance between the corresponding energy levels is $W = \sqrt{2hqB_0c^2} \sim 1.06\,(MeV)$, for $B_0 = 10^{10}\,(T)$. This ultra-relativistic particle promoted from the level with $n = 0$ to the one with $n = 1$ can produce, by de-excitation photons which will directly convert to electron-positron pairs.

Thus, one may compute only transitions between $n = 0$ and $n' = 1$, for which the characteristic values are given by

(26) $$\alpha_0 \approx \frac{b}{\kappa^2}\left\lfloor 1 - \frac{b}{2\kappa^2} \right\rfloor;\ \alpha_1 \approx \frac{b}{\kappa^2}\left\lfloor 3 - \frac{b}{2\kappa^2} \right\rfloor,$$

and one can find similar peaks and minima as in the figure 1.

For $\beta \to \infty$, the oscillatory behavior of the Mathieu functions happen in a shrinking neighborhood of $\kappa z = \zeta = \pi/2 + O(\beta^{-1/2})$. In explicit calculations, it is very useful to express the Mathieu function $MathieuC[\alpha_n, \beta, \zeta]$ in terms of parabolic cylinder functions and their derivatives, of variable $t = 2\sqrt{h} cos\zeta$, with $h = \sqrt{\beta}$, as [25]

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$$MathieuC \approx C_n \left\{ D_n(t) \sum_s \frac{A_s(t)}{h^s} + D'_n(t) \sum_s \frac{B_s(t)}{h^s} \right\},$$

where

$$D'_n = \frac{n}{2} D_{n-1} - \frac{1}{2} D_{n+1}, \ C_n \approx \left[\frac{\pi\sqrt{\beta}}{2(n!)^2} \right]^{1/4}$$

and the first terms of the polynomials $A_s(t)$ and $B_s(t)$ are

$$A_0 = 1, \quad A_1(t) = \frac{t^2}{2^6} = \frac{\sqrt{\beta}}{2^4} cos^2 \zeta,$$
$$B_1(t) = \frac{1}{2^5}\left[t^3 - (2n+1)t \right] \approx \frac{\beta^{3/4}}{4} cos^3 \zeta .$$

4. Conclusions

Within a perturbative approach, in the present paper, we are dealing with the Klein-Gordon equation, describing the bosons evolving in the (stationary) magnetar's crust, endowed with the electromagnetic configuration (2). The solution to the zero-order equation, expressed in terms of Bessel and Mathieu functions, is used for computing first-order transition amplitudes, for physically important ranges of model's parameters.

The width of the resonance bands are solely controlled by the parameter β in (11), which is characterizing the magnitude and the distribution of the magnetic field inside the crust.

A special attention is given to ultra-relativistic particles in ultra-strong magnetic field ($\beta > \alpha > 0$), when the particle momentum along *Oz* is satisfying the quantization law (24). The probability of the transition between two adiacent n values is discussed for different ranges of the model's parameters $b = qB_0$ and κ, characterizing the magnitude and the distribution of the magnetic field inside the crust. Thus, for $b/\kappa^2 < 2n+1$, a series of peaks, corresponding to high probabilities, can be noticed in the figure 1.

Finally, we mention that for particles moving in an ultra-strong magnetic field so that the α parameter defined in (11) is negative, the amplitude function is exponentially growing along *Oz* and there is only a narrow interval for the particle momentum p_z, i.e. $|\alpha| < \beta^2/2$, meaning

$$\frac{1}{2}\left(\frac{b}{\kappa}\right)^2 - \frac{1}{32\kappa^2}\left(\frac{b}{\kappa}\right)^4 < p_z^2 < \frac{1}{2}\left(\frac{b}{\kappa}\right)^2 \qquad (27)$$

which corresponds to a stable region in the Ince-Strutt diagram [26]. The present work can be extended in several directions, as for example to consider instead of the Klein-Gordon equation, the Dirac equation for relativistic fermions, leading to Mathieu's functions of complex variable and parameters [27].

These situations are very challenging and they are rarely discussed in literature. The existing packages written in MAPLE and MATHEMATICA are insufficient and some routines have been recently developed in [28]. It is worth to mention that the first study which considered a purely imaginary β parameter ($\beta = is, \ s \in \mathbb{R}$) was initiated by Mulholland and Goldstein [29] who deduced an approximative law governing the position of branching points in terms of a quadric growth. The authors detected the first branching point to be. $s_0 \approx 1.47$. For a more detailed analysis on the branching points we recommend [30, 31].

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