ON THE USE OF CONFORMING AND NONCONFORMING RECTANGULAR FINITE ELEMENTS FOR EIGENVALUE APPROXIMATIONS

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Abstract: *The paper deals with some combinations of conforming and nonconforming rectangular finite elements in order to obtain two-sided bounds of eigenvalues, applied to second-order elliptic opertor. The aim is to use the lowest possible order finite elements. Namely, the combination of serendipity conforming and rotated bilinear nonconforming elements is considered in details. This work continues some recent researches of the authors concerning eigenvalue approximations. Computational aspects of the used algorithm are also discussed. Finally, results from numerical experiments are presented.*

Keywords: RECTANGULAR FINITE ELEMENTS, CONFORMING/NONCONFORMING ELEMENTS, EIGENVALUE APPROXIMATION, TWO-SIDED BOUND

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1. Introduction and Preliminaries

We consider a bounded polygonal domain Ω in R^2 with boundary $\Gamma = \partial\Omega$. Given an integer $m \geq 0$, we use the Sobolev space $H^m(\Omega)$ with a norm $\|\cdot\|_{m,\Omega}$ and $(\cdot,\cdot)$ denotes the $L_2(\Omega)$ – inner product.

Consider the following weak form eigenvalue model problem: Find a number $\lambda \in R$ and a function $u \in V \equiv H_0^1(\Omega), \|u\|_{0,\Omega} = 1$ such that

$$a(u,v) = \lambda(u,v), \quad \forall v \in V, \tag{1}$$

where

$$a(u,v) = \iint_\Omega \nabla u \cdot \nabla v \, dx\, dy \quad \forall u,v \in V.$$

This problem has a countable sequence of real and positive eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq ...$ and the corresponding eigenfunctions $u_1, u_2, ...$ satisfy $(u_i, u_j) = \delta_{ij}, i,j \geq 1$.

Let $\{\tau_h\}$ be a family of a rectangulations of Ω which satisfies the usual regularity conditions (see [1]), i.e. there exists a constant $\sigma > 0$ such that $h_K / \rho_K \leq \sigma$, where h_K is the diameter of the rectangle $K \in \tau_h$ and ρ_K being the diameter of the largest circle contained in *K*. Then we denote $h = \max_{K \in \tau_h} h_K$.

So, we introduce a conforming finite element space $V_h \subset V$ based on the partition τ_h. Then, the corresponding approximation of (1) is: Find a number $\lambda_h \in R$ and a function $u_h \in V_h$, $\|u_h\|_{0,\Omega} = 1$ such that

$$a(u_h, v_h) = \lambda_h (u_h, v_h), \quad \forall v_h \in V_h. \tag{2}$$

For second- and fourth-order self-adjoint elliptic operator the eigenvalues computed using standard conforming finite element method (FEM) are always above the exact ones. This fact comes from minimum-maximum charachterization of the eigenvalues (see for example [2]). So that, it is an important and nontrivial problem to find methods giving lower bounds of the eigenvalues [3]. Herein, we claim a contribution to this specific field. More precisely, this paper is an extension of the authors' research done in [4] and [5].

First, let us consider nine-point conforming rectangular finite element. For any test function *v* and $K \in \tau_h$ the degrees of freedom which we will use are (see Fig. 1(a)) $v(a_j)$ and $\frac{1}{|l_j|}\int_{l_j} v(s)ds$ and $\frac{1}{|K|}\int_K v(x,y)\,dx\,dy$, where $a_j, j = 1,2,3,4$ are the vertices and $l_j, j = 1,2,3,4$ are the edges of *K*.

Then, the finite element space V_h is defined by (see [6]):

$$V_h = \{v \in H^1(\Omega) : v_{|K} = span\{1, x, y, xy, x^2, y^2, x^2y, xy^2, x^2y^2\}\}.$$

We can use also eight-point serendipity conforming finite element (see Fig. 1(b)). In this case:

$$V_h = \{v \in H^1(\Omega) : v_{|K} = span\{1, x, y, xy, x^2, y^2, x^2y, xy^2\}\}.$$

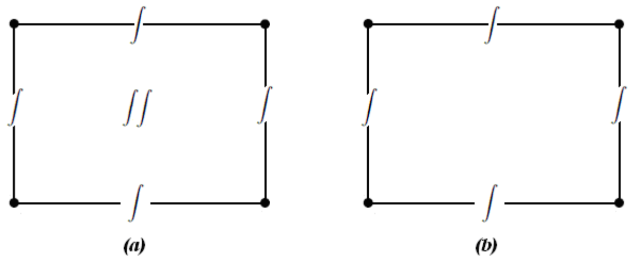

Fig. 1 *(a) 9-point rectangle; (b) 8-point serendipity rectangle*

Now we introduce non-conforming finite element spaces $\tilde{V}_h$ related to the same partition τ_h. We also introduce mesh-dependent bilinear form

$$a_h(u,v) = \sum_{K \in \tau_h} a_K(u,v), \; u,v \in V,$$

where

$$a_K(u,v) = \iint_K \nabla u \cdot \nabla v \, dx\, dy.$$

Obviously, in case of conforming FEM, $a(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$ coincide.

The space $\tilde{V}_h$ consists of Rannacher-Tourek (i.e. rotated bilinear) nonconforming finite elements denoted by Q_1^{rot} (Fig. 2(a)). Its degrees of freedom are the integral values at the edges of rectangle. An another case for construction of the space $\tilde{V}_h$ is to use the extension of Rannacher-Tourek element (so-called extended rotated bilinear element) denoted by EQ_1^{rot}. The degrees of freedom for this element are the integral values at the edges of the rectangle and the integral value over the rectangle (see Fig. 2(b)).

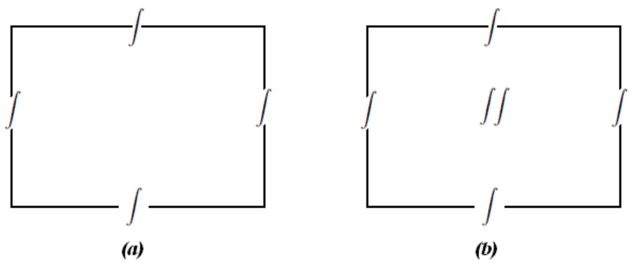

***Fig. 2** (a) Q_1^{rot} nonconforming element; (b) EQ_1^{rot} nonconforming element*

Let us define a so-called "nonconforming interpolation operator" $\tilde{i}_h : V \to \tilde{V}_h$ such that

$$a_h(v - \tilde{i}_h v, \tilde{v}_h) = 0, \ \forall v \in V, \tilde{v}_h \in \tilde{V}_h. \tag{3}$$

2. Main Result

Our result is based on determining a couple of finite element spaces $\left(V_h, \tilde{V}_h\right)$ which verify two sided bounds of eigenvalues λ using the following algorithm:

Algorithm

1. Find (λ_h, u_h) from (2) by means of conforming finite element space V_h;
2. Construct nonconforming space $\tilde{V}_h$ from V_h by eliminating some degrees of freedom in such a way that after obtaining of $\tilde{i}_h u_h \in \tilde{V}_h$ the condition (3) to be fulfilled;
3. Calculate the number $\tilde{\Lambda}_h = a_h(\tilde{i}_h u_h, \tilde{i}_h u_h)$ which approximates the exact eigenvalues λ asymptotically from below.

The case when V_h consists of nine-point quadratic conforming finite elements and $\tilde{V}_h$ contains EQ_1^{rot}-elements is proved by the authors in [4] (see Theorem 1). Later on, we will use the same notations as in [4].

Here, our aim is to prove the validity of the algorithm when V_h use the eight-point serendipity elements and $\tilde{V}_h$ -- Q_1^{rot}-nonconforming elements, respectively.

Thus the nonconforming finite element space is defined by (see [6]):

$$\tilde{V}_h = \left\{ v \in L_2(\Omega) : v_{|K} = span\left\{1, x, y, x^2 - y^2\right\}\right\}.$$

Also, for any $v \in V$:

$$\int_{l_j} \tilde{i}_h v(s)\, ds = \int_{l_j} v(s)\, ds, \tag{4}$$

with edges $l_j, j = 1,2,3,4$.

Introducing the notation $\left|v_h\right|_h^2 = a_h(v_h, v_h)$ for any $v_h \in V + \tilde{V}_h$, for our considerations, it is enough to assume the following interpolation inequality for $v \in V$:

$$\left|\tilde{i}_h v - v\right|_h \geq C h^{3/2}. \tag{5}$$

The following theorem contains a main result of the paper.

Theorem 1. Let (λ_h, u_h) be an approximation of the exact eigenpair (λ, u) obtained from (2) by means of eight-point serendipity elements. It the inequality (5) is valid, the number

$$\tilde{\Lambda}_h = a_h(\tilde{i}_h u_h, \tilde{i}_h u_h)$$

approximates λ from below when h is small enough, so that two-sided bounds of λ are obtained:

$$\tilde{\Lambda}_h \leq \lambda \leq \lambda_h. \tag{6}$$

Proof. We adopt the following notations for partial derivatives:

$$\partial_x \bullet = \frac{\partial \bullet}{\partial x}, \partial_y \bullet = \frac{\partial \bullet}{\partial y} \text{ and so on.}$$

Let $v_h \in \tilde{V}_h$ and $v \in V$. Then

$$\begin{aligned} a_h\left(\tilde{i}_h v - v, v_h\right) &= \sum_{K \in \tau_h} \iint_K \nabla\left(\tilde{i}_h v - v\right) \cdot \nabla v_h \, dx\, dy \\ &= \sum_{K \in \tau_h} \iint_K \left(\partial_x\left(\tilde{i}_h v - v\right)\partial_x v_h + \partial_y\left(\tilde{i}_h v - v\right)\partial_y v_h\right) dx\, dy. \end{aligned} \tag{7}$$

We choose K_0 to be a fixed reference rectangle. Then the result on arbitrary $K \in \tau_h$ will be transformed from K_0 using an affine transformation.

The equations of the edges of K_0 are:

$$l_{1;3} : y - y_0 = \mp \frac{h_2}{2};\ l_{2;4} : x - x_0 = \pm \frac{h_1}{2},$$

where (x_0, y_0) is the center of K_0 (Fig. 3) and $h = \sqrt{h_1^2 + h_2^2}$.

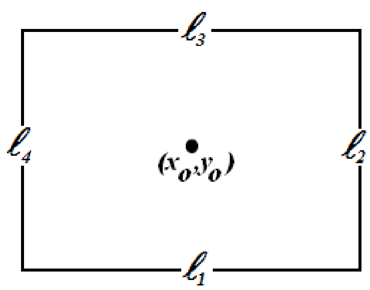

***Fig. 3** Rectangular reference element K_0*

Having in mind that $\tilde{V}_h$ is an incomplete biquadratic polynomial space on K_0, we can write

$$v_h(x,y)=v_h(x_0,y_0)+(x-x_0)\partial_x v_h(x_0,y_0)+(y-y_0)\partial_y v_h(x_0,y_0)$$

$$+\frac{1}{2}(x-x_0)^2\partial_{xx}v_h(x_0,y_0)+\frac{1}{2}(y-y_0)^2\partial_{yy}v_h(x_0,y_0).$$

Then obviously

$$\partial_x v_h(x,y)=\partial_x v_h(x_0,y_0)+(x-x_0)\partial_{xx}v_h(x_0,y_0),$$

$$\partial_y v_h(x,y)=\partial_y v_h(x_0,y_0)+(y-y_0)\partial_{yy}v_h(x_0,y_0).$$

Using Q_1^{rot}-element we have

$$\partial_{xx}v_h=-\partial_{yy}v_h=const. \qquad (8)$$

Applying the properties (4), from (7) we obtain

$$\iint_{K_0}\partial_x(\tilde{i}_h v-v)\partial_x v_h\,dx\,dy=\partial_x v_h(x_0,y_0)\left(\int_{l_2}-\int_{l_4}\right)(\tilde{i}_h v-v)dy$$

$$+h_1\partial_{xx}v_h\left(\int_{l_2}-\int_{l_4}\right)(\tilde{i}_h v-v)dy-\iint_{K_0}(\tilde{i}_h v-v)\partial_{xx}v_h\,dx\,dy$$

$$=-\iint_{K_0}(\tilde{i}_h v-v)\partial_{xx}v_h\,dx\,dy.$$

In the same manner,

$$\iint_{K_0}\partial_y(\tilde{i}_h v-v)\partial_y v_h\,dx\,dy=-\iint_{K_0}(\tilde{i}_h v-v)\partial_{yy}v_h\,dx\,dy.$$

Finally, from (8) it follows that

$$\iint_{K_0}\nabla(\tilde{i}_h v-v)\cdot\nabla v_h\,dx\,dy=0 \text{ for any } v_h\in\tilde{V}_h$$

and the relation (3) is proved for the element Q_1^{rot}.

Now, having in mind that $\|u_h\|_{0,\Omega}=1$, from (3) it follows

$$a_h(\tilde{i}_h u_h-u_h,\tilde{i}_h u_h-u_h)=a_h(\tilde{i}_h u_h,\tilde{i}_h u_h)-2a_h(\tilde{i}_h u_h,u_h)$$

$$+a_h(u_h,u_h)=a_h(u_h,u_h)-a_h(\tilde{i}_h u_h,\tilde{i}_h u_h)=\lambda_h-\tilde{\Lambda}_h,$$

consequently

$$\lambda_h-\tilde{\Lambda}_h=\left|\tilde{i}_h u_h-u_h\right|_h^2\geq 0.$$

Considering that λ_h is a conforming approximation of λ by means of serendipity rectangular finite element [2], we have

$$0\leq\lambda_h-\lambda\leq C\|u_h-u\|_{1,\Omega}^2\leq C_1\left(h^{2-\delta}\right)^2,$$

where δ is a small positive number.

Moreover, from (5) we obtain asymptotically the inequality

$$\lambda-\tilde{\Lambda}_h=\lambda-\lambda_h+\lambda_h-\tilde{\Lambda}_h=-(\lambda_h-\lambda)+\left|\tilde{i}_h u_h-u_h\right|_h^2$$

$$\geq -C_1 h^{4-2\delta}+C_2 h^3\geq 0.$$

Thus, (6) is proved.

■

3. Numerical Results

The numerical experiments and the resulting approximations of eigenvalues given in this section verify and confirm the validity, reliability and effectiveness of the proposed algorithm.

For purpose of demonstration of the method we propose, we solve the problem (1) on square domain $\Omega=[0,\pi]\times[0,\pi]$.

This choice we have made because of the fact that in this case the exact eigenvalues are known and are equal to $s_1^2+s_2^3$, $s_1,s_2=1,2,3,...$ So that, $\lambda_1=2;\lambda_2=5;\lambda_3=5;...$

For our numerical implementation we divide the domain Ω into N^2 squares, *N = 4; 8; 12; 16; 20*. Thus the mesh parameter *h* is equal to $\frac{\pi\sqrt{2}}{N}$.

We solve the variational discrete problem (2) using conforming quadratic rectangular (9-point) finite elements for V_h (see Fig. 1(a)). As a result, we obtain the approximate eigenvalues λ_h, which give upper bounds for the corresponding exact eigenvalues and the approximate eigenfunctions u_h which we interpolate. Interpolation is done by means of EQ_1^{rot}-nonconforming finite element space (Fig. 2(b)). Obtaining the intepolants $\tilde{i}_h u_h\in\tilde{V}_h$ and calculating the numbers $\tilde{\Lambda}_h=a_h(\tilde{i}_h u_h,\tilde{i}_h u_h)$ we get lower bounds for the exact eigenvalues. The results from our numerical experiment for the first three eigenvalues are give in Table 1 and Table 2, respectively.

Table 1: *Approximations of the exact eigenvalues obtained after finite element implementation by means of 9-point conforming rectangular quadratic finite elements*

	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$
N = 4	2.001024281	5.030616274	5.032574099
N = 8	2.000065532	5.002110541	5.004469411
N = 12	2.000013002	5.000450150	5.002890554
N = 16	2.000004120	5.000165987	5.0026355170
N = 20	2.000001689	5.000087946	5.002571064
Exact eigenvalues	2	5	5

Table 2: *Approximations of the exact eigenvalues obtained after nonconforming EQ_1^{rot} − interpolation*

	$\tilde{\Lambda}_{1,h}$	$\tilde{\Lambda}_{2,h}$	$\tilde{\Lambda}_{3,h}$
N = 4	1.902219920	4.655142682	4.656420455
N = 8	1.974624003	4.901823878	4.903948840
N = 12	1.988641719	4.955268267	4.957598338
N = 16	1.993595054	4.974626877	4.977033035
N = 20	1.995896103	4.983705600	4.986147765
Exact eigenvalues	2	5	5

Next, for the same model problem we demonstrate the efficiency of the proposed algorithm for more simple finite elements.

We solve the variational discrete problem (2) using eight-point serendipity conforming finite elements for V_h (Fig. 1(b)). As a result we obtain the approximate eigenvalues λ_h giving upper bounds for the corresponding exact eigenvalues and the approximate eigenfunctions u_h which we interpolate. Interpolation is done by means of Q_1^{rot}-nonconforming finite element space (Fig. 2(a)). Obtaining the intepolants $\tilde{i}_h u_h \in \tilde{V}_h$ and calculating the numbers $\tilde{\Lambda}_h = a_h(\tilde{i}_h u_h, \tilde{i}_h u_h)$ we get lower bounds for the exact eigenvalues. The results from our numerical experiment for the first three eigenvalues are give in Table 3 and Table 4, respectively.

Table 3: *Approximations of the exact eigenvalues obtained after finite element implementation by means of 8-point serendipity conforming rectangular finite elements*

	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$
N = 4	2.001091866	5.032026684	5.032037751
N = 8	2.0000664317	5.002126594	5.004493905
N = 12	2.000066432	5.000451611	5.002891734
N = 16	2.000004134	5.000166314	5.002635721
N = 20	2.000001692	5.000088072	5.002571117
Exact eigenvalues	2	5	5

Table 4: *Approximations of the exact eigenvalues obtained after nonconforming Q_1^{rot} – interpolation*

	$\tilde{\Lambda}_{1,h}$	$\tilde{\Lambda}_{2,h}$	$\tilde{\Lambda}_{3,h}$
N = 4	1.881513965	4.124185537	4.124191017
N = 8	1.949450862	4.748511061	4.750459845
N = 12	1.977322992	4.885339250	4.887582856
N = 16	1.987202566	4.934933147	4.937289058
N = 20	1.999994922	4.979244596	4.980117042
Exact eigenvalues	2	5	5

The approximate values in Table 1 and Table 3 are greater than the exact ones and the sequences $\{\lambda_{j,h}\}, j=1,2,3$ obtained when the mesh parameter *h* decreases are decreasing. This is a reasonable and expected numerical result, because of the fact that conforming finite element methods are used [2].

On the part of the nonconforming interpolation implementation, from the approximations of the eigenvalues given in Table 2 and Table 4, respectively, it is clear that the resulting approximation of the exact eigenvalues is from below. When the mesh parameter *h* decreases, the sequences $\{\tilde{\Lambda}_{j,h}\}, j=1,2,3$ are increasing and go to the corresponding exact eigenvalue.

As it is well-known from the theoretical point of view, nine-point and eight-point conforming finite elements give similar results concerning the error estimates (Table 1 and Table 3). As a consequence of this, as well as from the fact that Q_1^{rot} and EQ_1^{rot}-nonconforming finite elements give one and the same convergence order, the results in Table 2 and Table 4 are also similar.

3. Conclusions

The proposed algorithm gives lower and upper bounds of the eigenvalues simultaneously – only to solve by conforming FEM the eigenvalue problem once and additionally to apply a nonconforming interpolation to the conforming solution is needed.

In this paper use of more simple and convenient finite elements is proposed, proved theoretically and demonstrated.

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