# A MATHEMATICAL MODEL OF VISCOUS LIQUID MIXTURE MOTION THROUGH A VERTICAL CYLINDRICAL PIPE

Asst., MSc Sorokina Natalia,
Institute of Computer Science and Technology – Peter the Great Saint-Petersburg Polytechnic University, Russia
snv_special@inbox.ru

***Abstract:*** In the paper a mathematical model of the non-stationary motion of a viscous liquid mixture through the vertical straight pipe of the circular cross section is proposed. During the model construction weak compressibility of the mixture is considered. The Navier-Stokes equations system is taken as a basis. Such model can be used in the description of oil motion in a vertical well.

**Keywords**: NON-STATIONARY HYDRODYNAMICS, LIQUID MIXTURES, WEAK COMPRESSIBILITY, VERTICAL PIPE, MATH MODELING

## 1. Introduction

The problem of the liquid mixtures (or emulsions, suspensions) motion is not just interesting, but also very important in some areas of the national economy, such as oil and gas industry. It is well known that during the extraction of oil it is not a “clear” product ascending from under the ground, but a mixture, consisting of oil, water and gas. In other words, we deal with the water-oil emulsion.

As emulsion move up through the tubing its properties, such as density and viscosity, are changing. Its change is connected with temperature and pressure. And it is important to know the velocity of mixture’s ascending, of course.

Further in the paper we use the term “liquid” instead of “liquid mixture” since we assume that this mixture is homogeneous – its components are distributed evenly in the main phase, they are well mixed. This is the first assumption. We have to obtain the mathematical model, describing the non-stationary motion of the weakly compressible liquid through the vertical pipe of the circular cross-section.

## 2. The basic equations

We take the equations of continuum mechanics as the basis [1], consider that there are no sources (or drains on the contrary) of the mass:

$$\frac{d\rho}{dt}+\rho\left(\vec{\nabla}\cdot\vec{\upsilon}\right)=0, \qquad (1)$$

$$\rho\frac{d\vec{\upsilon}}{dt}=\rho\vec{F}+\frac{\partial\vec{\tau}_x}{\partial x}+\frac{\partial\vec{\tau}_y}{\partial y}+\frac{\partial\vec{\tau}_z}{\partial z}, \qquad (2)$$

$$\rho\frac{d\vec{M}}{dt}=\rho\vec{\Pi}+\vec{i}\times\vec{\tau}_x+\vec{j}\times\vec{\tau}_y+\vec{k}\times\vec{\tau}_z+\frac{\partial\vec{\pi}_x}{\partial x}+\frac{\partial\vec{\pi}_y}{\partial y}+\frac{\partial\vec{\pi}_z}{\partial z}, \qquad (3)$$

$$\rho\frac{dE}{dt}=\varepsilon+\vec{\tau}_x\cdot\frac{\partial\vec{\upsilon}}{\partial x}+\vec{\tau}_y\cdot\frac{\partial\vec{\upsilon}}{\partial y}+\vec{\tau}_z\cdot\frac{\partial\vec{\upsilon}}{\partial z}+\vec{\nabla}\cdot\vec{t}. \qquad (4)$$

Since we consider vertical motion, which is obviously due to the head from below, we assume that rotary motion in the liquid is absent hence we will not need the equation (3) further. As the first approximation we may assume that neither density nor pressure depends on temperature and hence energy change may be ignored. That allows us to neglect the equation (4).

## 3. The construction of the mathematical model

We introduce the Cartesian coordinate system and direct the axis $Oz$ along the pipe axis. We write the system of equations (1)-(2) in projections onto the Cartesian axes:

$$\frac{\partial\rho}{\partial t}+\upsilon_x\frac{\partial\rho}{\partial x}+\upsilon_y\frac{\partial\rho}{\partial y}+\upsilon_z\frac{\partial\rho}{\partial z}+\rho\left(\frac{\partial\upsilon_x}{\partial x}+\frac{\partial\upsilon_y}{\partial y}+\frac{\partial\upsilon_z}{\partial z}\right)=0, \qquad (5)$$

$$\frac{\partial\upsilon_x}{\partial t}+\upsilon_x\frac{\partial\upsilon_x}{\partial x}+\upsilon_y\frac{\partial\upsilon_x}{\partial y}+\upsilon_z\frac{\partial\upsilon_x}{\partial z}=F_x+\frac{1}{\rho}\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}\right), \qquad (6)$$

$$\frac{\partial\upsilon_y}{\partial t}+\upsilon_x\frac{\partial\upsilon_y}{\partial x}+\upsilon_y\frac{\partial\upsilon_y}{\partial y}+\upsilon_z\frac{\partial\upsilon_y}{\partial z}=F_y+\frac{1}{\rho}\left(\frac{\partial\tau_{yx}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}\right), \qquad (7)$$

$$\frac{\partial\upsilon_z}{\partial t}+\upsilon_x\frac{\partial\upsilon_z}{\partial x}+\upsilon_y\frac{\partial\upsilon_z}{\partial y}+\upsilon_z\frac{\partial\upsilon_z}{\partial z}=F_z+\frac{1}{\rho}\left(\frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{zy}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}\right). \qquad (8)$$

Since it is a vertical pipe, the gravity acts on the liquid (per the elementary volume): $F_x=0$, $F_y=0$, $F_z=-\rho g$. Next suppose that velocity has only the vertical component, in other words $\upsilon_x=\upsilon_y=0$. In p.2 it is said we consider that density does not depend on temperature. Let us clarify this assumption: let temperature vary with the altitude of the rise, i.e. with the change of the $z$ coordinate, and since the density depends on temperature (also on pressure) we have an implicit density dependence on $z$ coordinate: $\rho=\rho\left(T(z)\right)=\rho(z)$.

We write the system (5)-(8) with the assumptions described:

$$\upsilon_z\frac{\partial\rho}{\partial z}+\rho\frac{\partial\upsilon_z}{\partial z}=0, \qquad (5')$$

$$\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}=0, \qquad (6')$$

$$\frac{\partial\tau_{yx}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}=0, \qquad (7')$$

$$\frac{\partial\upsilon_z}{\partial t}+\upsilon_z\frac{\partial\upsilon_z}{\partial z}=-\rho g+\frac{1}{\rho}\left(\frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{zy}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}\right). \qquad (8')$$

In general the stress tensor components have the following form [1]:

$$\tau_{xx}=-p+\lambda\left(\frac{\partial\upsilon_x}{\partial x}+\frac{\partial\upsilon_y}{\partial y}+\frac{\partial\upsilon_z}{\partial z}\right)+2\mu\frac{\partial\upsilon_x}{\partial x}, \qquad (9)$$

$$\tau_{yy}=-p+\lambda\left(\frac{\partial\upsilon_x}{\partial x}+\frac{\partial\upsilon_y}{\partial y}+\frac{\partial\upsilon_z}{\partial z}\right)+2\mu\frac{\partial\upsilon_y}{\partial y}, \qquad (10)$$

$$\tau_{zz}=-p+\lambda\left(\frac{\partial\upsilon_x}{\partial x}+\frac{\partial\upsilon_y}{\partial y}+\frac{\partial\upsilon_z}{\partial z}\right)+2\mu\frac{\partial\upsilon_z}{\partial z} \qquad (11)$$

$$\tau_{xy}=\tau_{yx}=\mu\left(\frac{\partial\upsilon_x}{\partial y}+\frac{\partial\upsilon_y}{\partial x}\right),\ \tau_{yz}=\tau_{zy}=\mu\left(\frac{\partial\upsilon_y}{\partial z}+\frac{\partial\upsilon_z}{\partial y}\right), \qquad (12)$$
$$\tau_{zx}=\tau_{xz}=\mu\left(\frac{\partial\upsilon_x}{\partial z}+\frac{\partial\upsilon_z}{\partial x}\right).$$

According to the assumptions made, the stress tensor components take the following form:

$$\tau_{xx}=-p+\lambda\frac{\partial\upsilon_z}{\partial z}, \qquad (9')$$

$$\tau_{yy}=-p+\lambda\frac{\partial\upsilon_z}{\partial z}, \qquad (10')$$

$$\tau_{zz} = -p + [\lambda + 2\mu]\frac{\partial \upsilon_z}{\partial z} \quad (11')$$

$$\tau_{xy} = \tau_{yx} = 0, \tau_{yz} = \tau_{zy} = \mu\frac{\partial \upsilon_z}{\partial y}, \tau_{zx} = \tau_{xz} = \mu\frac{\partial \upsilon_z}{\partial x}. \quad (12')$$

Substitute these relations in the system ($5'$)–($8'$):

$$\upsilon_z \frac{\partial \rho}{\partial z} + \rho\frac{\partial \upsilon_z}{\partial z} = 0, \quad (5')$$

$$-\frac{\partial p}{\partial x} + [\lambda + \mu]\frac{\partial^2 \upsilon_z}{\partial x \partial z} = 0, \quad (6')$$

$$-\frac{\partial p}{\partial y} + [\lambda + \mu]\frac{\partial^2 \upsilon_z}{\partial y \partial z} = 0, \quad (7')$$

$$\frac{\partial \upsilon_z}{\partial t} + \upsilon_z\frac{\partial \upsilon_z}{\partial z} = -\rho g + \frac{1}{\rho}\left(\mu\frac{\partial^2 \upsilon_z}{\partial x^2} + \mu\frac{\partial^2 \upsilon_z}{\partial y^2} + \lambda\frac{\partial^2 \upsilon_z}{\partial z^2} - \frac{\partial p}{\partial z}\right). \quad (8')$$

Let us turn to the cylindrical coordinates in the system ($5'$)–($8'$), assuming in advance the cylindrical symmetry of the flow:

$$\upsilon_z \frac{\partial \rho}{\partial z} + \rho\frac{\partial \upsilon_z}{\partial z} = 0, \quad (5')$$

$$[\lambda + \mu]\cos\varphi\frac{\partial^2 \upsilon_z}{\partial r \partial z} - \cos\varphi\frac{\partial p}{\partial r} = 0, \quad (6')$$

$$[\lambda + \mu]\sin\varphi\frac{\partial^2 \upsilon_z}{\partial r \partial z} - \sin\varphi\frac{\partial p}{\partial r} = 0, \quad (7')$$

$$\frac{\partial \upsilon_z}{\partial t} + \upsilon_z\frac{\partial \upsilon_z}{\partial z} = -\rho g + \frac{1}{\rho}\left(\mu\left[\frac{\partial^2 \upsilon_z}{\partial r^2} + \frac{1}{r}\frac{\partial \upsilon_z}{\partial r}\right] + \lambda\frac{\partial^2 \upsilon_z}{\partial z^2} - \frac{\partial p}{\partial z}\right). \quad (8')$$

Next we add two equations ($6'$) and ($7'$):

$$[\lambda + \mu][\cos\varphi + \sin\varphi]\left(\frac{\partial^2 \upsilon_z}{\partial r \partial z} - \frac{\partial p}{\partial r}\right) = 0.$$

Since sum of bulk and dynamic viscosity coefficients cannot be zero, sum of sine and cosine of the same angle cannot be zero, it follows that

$$\frac{\partial^2 \upsilon_z}{\partial r \partial z} - \frac{\partial p}{\partial r} = 0,$$

or

$$\frac{\partial^2 \upsilon_z}{\partial r \partial z} = \frac{\partial p}{\partial r}. \quad (13)$$

Thus the final system of equations has the following form (we omit the $z$ index since we assume that there is only one velocity component):

$$\upsilon \frac{\partial \rho}{\partial z} + \rho\frac{\partial \upsilon}{\partial z} = 0, \quad (5')$$

$$\frac{\partial^2 \upsilon}{\partial r \partial z} = \frac{\partial p}{\partial r}, \quad (13)$$

$$\frac{\partial \upsilon}{\partial t} + \upsilon\frac{\partial \upsilon}{\partial z} = -\rho g + \frac{1}{\rho}\left(\mu\left[\frac{\partial^2 \upsilon}{\partial r^2} + \frac{1}{r}\frac{\partial \upsilon}{\partial r}\right] + \lambda\frac{\partial^2 \upsilon}{\partial z^2} - \frac{\partial p}{\partial z}\right). \quad (8')$$

Three equations contain three unknown functions: density $\rho(z)$, pressure $p(z)$ and velocity $\upsilon(r,z,t)$.

## *4. The formulation of the corresponding problem and its correctness*

As noted earlier, the problem of studying the motion of a viscous liquid (liquid mixture) through the vertical pipe arises in the field of oil products extraction. It is necessary to predict the pressure, viscosity and velocity of the mixture that rises from the depth along the pipe, to understand whether, for example, pressure changes are so critical that it will lead to partial equipment destruction (the same pipe).

For the mathematical formalization of the corresponding problem, it is necessary to set the initial pressure at the pipe inlet and pipe outlet (at the pipe outlet it is possibly constant), initial velocity distribution over the cross-section and initial density of the liquid. In time, or, in the first approximation, during the ascending along the change in pressure the liquid density changes, since the saturation of the mixture with gas increases [4,5].

In solving partial differential equations (systems of equations), the study of the correctness of the corresponding initial-boundary value problems for equations being solved is especially important. For quasilinear partial differential equations of the second order this analysis is not simple. For the system of equations ($5'$), (13), ($8'$), such a study was carried out by OA. Ladyzhenskaya [2]. The results are positive.

## *5. The solution method*

For the initial-boundary value problem, the system of equations obtained in p.3 can be solved by a numerical method – the variable direction method [3]. According to this method, the previously obtained partial differential equations are replaced by difference relations with intermediate computation of functions on the $k+0.5$ -th time layer: A similar solution has already been applied in [4].

## *6. Conclusion*

In this paper, the construction of a mathematical model describing the nonstationary motion of a viscous compressible mixture over a long vertical rectilinear tube of circular cross-section is considered. Such a model can be used to solve the problem of the movement of oil along a vertical well. A numerical method for solving the corresponding initial-boundary value problem is proposed.

## *7. References*

[1] Vallander, S.V. Lectures on fluid dynamics. – L.: Izd-vo Leningr. un-ta, 1973, 296 p. (in Russian)

[2] Ladyzhenskaya, O. A. On the unique solvability in the large of the three-dimensional Cauchy problem for the Navier-Stokes equations in the presence of axial symmetry (in Russian), Zapiski Nauchnikh Seminarov LOMI, 1968, v. 7, p. 155-177

[3] Filatov, E. Y., Yasinskiy, F. N. Mathematical modeling of liquid and gas flows: Tutorial / State Educational Institution of Higher Professional education "Ivanovo Power Engineering Institute of V. I. Lenin". – Ivanovo, 2007 (in Russian)

[4] Firsov A., Sorokina N. Mathematical modeling of non-stationary flows of liquid homogeneous viscous mixtures by pipelines // Mathematical modeling. – 2017. – Issue 1/2017. – P.18-22

[5] Islamov M.K. Borehole oil extraction [Electronic resource]: Theoretical basis for lifting fluid and gas from a well URL: http://doidpo.rusoil.net/storage/Downhole%20oil%20production%28Islamov%29/to/to4_1.html (access date: 21.11.17) (in Russian)

[6]Everything about oil [Electronic resource]: Oil extraction. Methods of operation of wells. URL: http://vseonefti.ru/upstream/sposoby-dobychi.html (access date: 21.11.17) (in Russian)