STABILITY ANALYSIS AND SIMULATIONS OF BIOREACTOR MODEL WITH DELAYED FEEDBACK

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Abstract: *We consider a well known mathematical model of continuous methane fermentation, consisting of two nonlinear ordinary differential equations and one algebraic equation for the gaseous output. The model involves one microbial population and one substrate. We propose an output feedback including a discrete delay and use it for asymptotic stabilization of the model. Feedback control of bioreactor models provides many advantages in operating a plant, mainly by increasing its efficiency. We also propose a numerical model-based extremum seeking algorithm for maximizing the biogas (methane) flow rate in real time. Numerical simulations using this algorithm are included. The simulations are implemented in the Python programming language which is recently recognized as a powerful modern general purpose object-oriented language.*

Keywords: BIOREACTOR MODEL, DELAY FEEDBACK CONTROL, EXTREMUM SEEKING ALGORITHM

1. Introduction

We consider a well known mathematical model of the continuous methane fermentation, consisting of two nonlinear ordinary differential equations and one algebraic equation

$$\begin{aligned} \frac{ds}{dt} &= -k_1\mu(s)x + u(s_{in} - s) \\ \frac{dx}{dt} &= (\mu(s) - \alpha u)x \end{aligned} \tag{1}$$

with gaseous output

$$Q(s,x) = k_2\,\mu(s)x, \tag{2}$$

where $x = x(t)$ and $s = s(t)$ are state variables, x is biomass concentration [g/dm^3], s – substrate concentration [g/dm^3], u – dilution rate [day $^{-1}$], s_{in} – influent substrate concentration [g/dm^3], k_1 – yield coefficient [-], k_2 – coefficient [(dm^3)^2/g], Q – methane gas flow rate [dm^3/day]. The parameter $\alpha \in (0,1)$ represents the proportion of bacteria that are affected by the dilution; $\alpha = 0$ and $\alpha = 1$ correspond to an ideal fixed bed reactor and to an ideal continuous stirred tank reactor, respectively. The dilution rate u is considered as a control variable. The model function $\mu(s)$ represents the specific growth rate of the biomass.

Assumption A1. We assume that μ is defined for $s \in [0,+\infty)$, $\mu(0) = 0$ and $\mu(s) > 0$ whenever $s > 0$, $\mu(s)$ is continuously differentiable for all $s > 0$.

We propose here an output feedback including a discrete delay and use it for asymptotic stabilization of the dynamic model (1). Feedback control of bioreactor models provides many advantages in operating a plant and is used to increase its efficiency. On the other hand, there is always a time delay between input and output measurements in industrial and biological systems [4].

We make the following assumption.

Assumption A2. Assume that lower bounds s_{in}^- and k_2^- for the values of s_{in} and k_2 respectively, and an upper bound k_1^+ for the value of k_1 are known.

Denote for simplicity

$$\beta^- = \frac{k_1^+}{k_2^-\, s_{in}^-}.$$

Define the following feedback control law:

$$k(s,x) = \beta k_2\mu(s)x \text{ with } \beta \in (\beta^-,+\infty).$$

Replacing in the model (1)–(2) u by the feedback

$$\kappa(s(t-\tau),\ x(t-\tau)),$$

where $\tau > 0$ is a discrete delay, we obtain the following control system:

$$\begin{aligned} \frac{ds}{dt} &= -k_1\mu\big(s(t)\big)x(t) \\ &\quad +\beta k_2\mu\big(s(t-\tau)\big)x(t-\tau)\big(s_{in} - s(t)\big) \end{aligned} \tag{3}$$

$$\begin{aligned} \frac{dx}{dt} &= \mu\big(s(t)\big)x(t) \\ &\quad -\alpha\beta k_2\mu\big(s(t-\tau)\big)x(t-\tau) \end{aligned} \tag{4}$$

Choose some $\beta \in (\beta^-,+\infty)$ and let

$$\bar{s} := s_{in} - \frac{k_1}{k_2\beta} \tag{5}$$

Obviously, $\bar{s}$ belongs to the interval $(0, s_{in})$. Define

$$\bar{x} := \frac{1}{\alpha\beta k_2} \tag{6}$$

It is straightforward to see that the point

$$\bar{p}_\beta := (\bar{s},\bar{x})$$

is an equilibrium point for (3)–(4).

2. Stability analysis of the model

Assumption A3. We assume that the following inequalities hold true

$$\mu(s^-) < \mu(\bar{s}) < \mu(s^+)$$

for each $s^- \in (0,\bar{s})$ and $s^+ \in (\bar{s}, s_{in})$.

Assumption A3 is technical. It is always fulfilled when the function $\mu(s)$ is monotone increasing (like the Monod specific growth rate, see Section 4). If the function $\mu(s)$ is not monotone increasing (like e. g. the Haldane law) then the point $\bar{s}$ (i.e. β) has to be chosen in a proper way in order to satisfy Assumption A3.

Below we present a result about the local asymptotic stability of the equilibrium point $\bar{p}_\beta$ with respect to the parameters of the closed-loop system (3)–(4). Denote for simplicity:

$$\begin{aligned} b &= k_1\bar{x}\mu(\bar{s})\mu'(\bar{s}) \\ d &= \mu(\bar{s})\left(\frac{1}{\alpha}\mu(\bar{s}) - k_1\bar{x}\mu(\bar{s})\right) \end{aligned}$$

where the coefficients b and d depend on the parameter β, since $\bar{s}$ and $\bar{x}$ are functions of β, see (5) and (6).

Theorem 1.

(i) If $b \geq d$ then the equilibrium point $\bar{p}_\beta$ is locally asymptotically stable for any value of the delay $\tau > 0$.

(ii) If $b < d$ then there exists a delay $\tau_0 > 0$ such that the equilibrium point $\bar{p}_\beta$ is locally asymptotically stable for all values τ of the delay such that $\tau < \tau_0$; the equilibrium is locally unstable if $\tau \geq \tau_0$, and a Hopf bifurcation occurs at $\tau = \tau_0$.

The proof of the theorem is rather long and will not be presented here due to the restricted paper length. The proof is based on some ideas from [3], [5] and [6].

We suppose that in the cases when the equilibrium point $\bar{p}_\beta$ is locally asymptotically stable then it is also globally asymptotically stable. The computer simulations confirm that assumption (see Section 4). Proving global stability of $\bar{p}_\beta$ will be a subject of further studies.

3. Maximizing the biogas production via extremum seeking

Consider the equation (2) describing the process output, i.e. the methane (biogas) production. In this section we shall shortly present a numerical extremum seeking algorithm (ESA), cf. [1]–[2], to steer and stabilize the dynamics (3)–(4) towards a steady state, where maximum methane flow rate $\boldsymbol{Q_{max}}$ is achieved. For that purpose we compute $\boldsymbol{Q}$ on the set of all equilibrium points, parameterized with respect to $\boldsymbol{\beta}$. Denote the so obtained function by $\boldsymbol{Q(\beta)}$, where $\boldsymbol{\beta} \in (\boldsymbol{\beta}^-, +\infty)$. The function $\boldsymbol{Q(\beta)}$ is called input-output static characteristic of the model. Assume that the function $\boldsymbol{\beta} \to \boldsymbol{Q(\beta)}$, $\boldsymbol{\beta} \in (\boldsymbol{\beta}^-, +\infty)$, is strongly unimodal, i.e. there exists a unique point $\boldsymbol{\beta_{max}} \in (\boldsymbol{\beta}^-, +\infty)$ where $\boldsymbol{Q(\beta)}$ takes a maximum, $\boldsymbol{Q_{max}} = \boldsymbol{Q(\beta_{max})}$, the function strongly increases in the interval $(\boldsymbol{\beta}^-, \boldsymbol{\beta_{max}})$ and strongly decreases in $(\boldsymbol{\beta_{max}}, +\infty)$.

Let

$$\boldsymbol{p_{\beta_{max}}} = (\bar{\boldsymbol{s}}_{\boldsymbol{max}}, \bar{\boldsymbol{x}}_{\boldsymbol{max}})$$

be the steady state where Q_{max} is achieved. Our goal is to stabilize in real time the system (3)–(4) towards this (unknown) equilibrium point $p_{\beta_{max}}$ and therefore to the maximum methane flow rate Q_{max}.

The main idea of ESA is the following: we construct a sequence of points $\beta^1, \beta^2, \ldots, \beta^n, \ldots$ from the interval $(\beta^-, +\infty)$, each β^j being in the form $\beta^j = \beta^{j-1} \pm h,\ h > 0$, and such that the sequence $\{\beta^j\}$ tends to β_{max}. Then by computing and comparing the values $Q(\beta^1), Q(\beta^1), \ldots, Q(\beta^n), \ldots$, we achieve the desired equilibrium point $p_{\beta_{max}}$ and thus Q_{max}.

In the computer implementation ESA is carried out in two stages. In the first stage, "rough" intervals $[\beta]$ and $[Q]$ are found which enclose β_{max} and Q_{max} respectively; in the second stage, the interval $[\beta]$ is refined using an elimination procedure based on the golden mean value (or Fibonacci search) strategy. The second stage produces the final intervals $[\beta_{max}] = [\beta_{max}^-, \beta_{max}^+]$ and $[Q_{max}]$ such that $\beta_{max} \in [\beta_{max}]$, $Q_{max} \in [Q_{max}]$ and $\beta_{max}^+ - \beta_{max}^- \leq \varepsilon$, where the tolerance $\varepsilon > 0$ is specified by the user.

The ESA was firstly developed for the model (3)–(4) with $\tau = 0$, cf. [1], [2]. Now the algorithm is modified for the delayed model and is implemented in the object-oriented language *Python*. Nowadays, this programming language has become an integral part of scientific and engineering computing due to the vast abundance of open source libraries and interfaces to major software tools.

4. Numerical Simulation

In the simulation process we consider the Monod specific growth rate

$$\mu(s) = \frac{m_1 s}{k_s + s},$$

where m_1 is the maximum specific growth rate of the microorganisms [1/day] and k_s is the saturation constant [g/dm^3].

We use the following numerical values for the model parameters (cf. [1] and the references therein):

$$k_1 = 27.4,\ s_{in} = 3,\ m_1 = 0.4,\ k_s = 0.4,\ \alpha = 0.5,\ k_2 = 75.$$

With $k_1^+ = 27.6$, $s_{in}^- = 2.8$ and $k_2^- = 74.8$ we obtain that $\beta^- = \frac{k_1^+}{s_{in}^- k_2^-} \approx 0.1318$. Therefore the parameter β varies in the interval $(0.1318, +\infty)$.

For the above parameter values, the input-output static characteristic $Q(\beta)$ is strongly unimodal and takes its maximum for $\beta = \beta_{max} = 0.16355$ (see Figure 1), thus $Q_{max} = 3.21376$, $\bar{s}_{max} = 0.76619$ and $\bar{x}_{max} = 0.16305$.

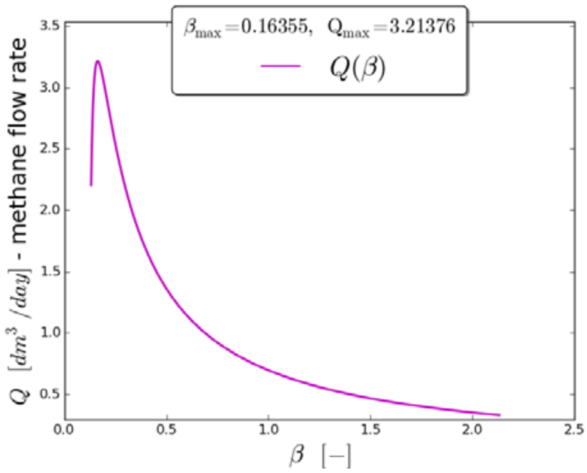

Fig. 1. *The input-output static characteristic* $Q(\beta)$

4.1. Simulation of the global dynamics behavior

First we shall demonstrate numerically the stability of the system (3)–(4) towards the equilibrium point $\bar{p}_\beta$ for different values of the delay τ and the parameter β.

(i) $\beta = \beta_{max} = 0.16355$

For this value of β, the inequality $b > d$ is fulfilled. According to Theorem 1, the equilibrium point $(\bar{s}_{max}, \bar{x}_{max}) = (0.76619, 0.16305)$ is stable for any $\tau \geq 0$ as it can be seen in the Figures 2 to 4 for the following values of the delay: $\tau = 1$, $\tau = 5$ and $\tau = 15$.

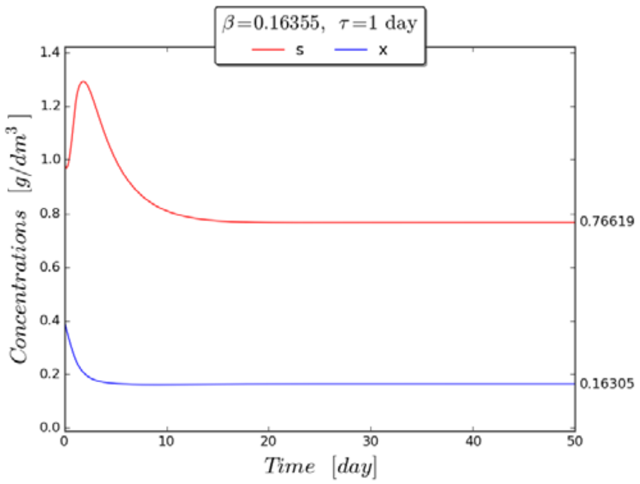

Fig. 2. $\beta = \beta_{max} = 0.16355$, $\tau = 1$:
Time evolution of s *and* x *towards* $\bar{s}_{max}$ *and* $\bar{x}_{max}$ *respectively*

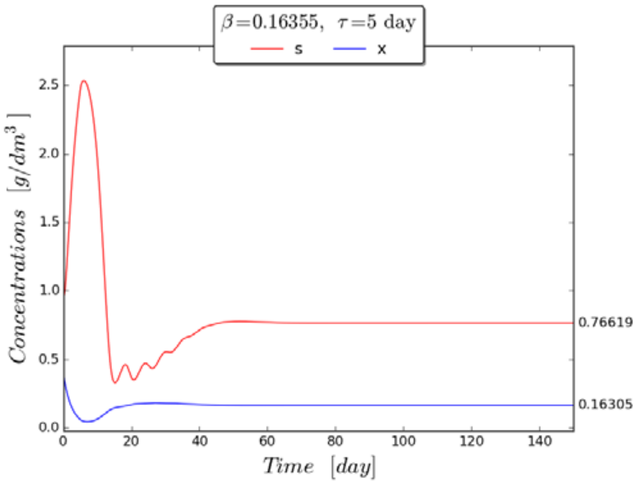

Fig. 3. $\beta = \beta_{max} = 0.16355$, $\tau = 5$: *Time evolution of* s *and* x *towards* $\bar{s}_{max}$ *and* $\bar{x}_{max}$ *respectively*

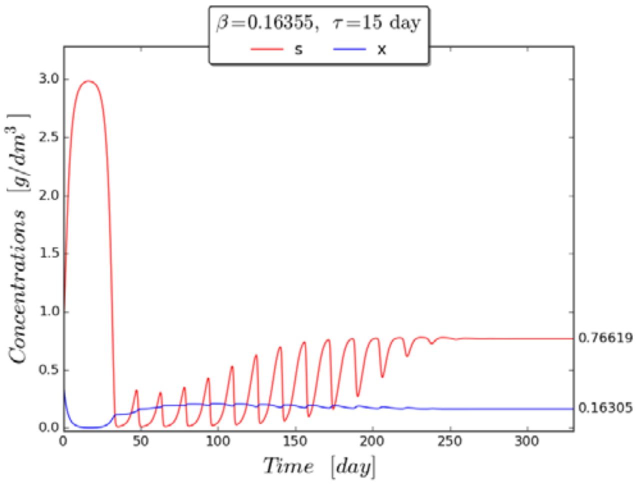

Fig. 4. $\beta = \beta_{max} = 0.16355$, $\tau = 15$: *Time evolution of* s *and* x *towards* $\bar{s}_{max}$ *and* $\bar{x}_{max}$ *respectively*

(ii) $\beta = 1.0$

Here $b < d$ is valid. In this case $\tau_0 = 4.70326$, $\bar{s} = 2.63467$ and $\bar{x} = 0.02667$. According to Theorem 1, the equilibrium point $(\bar{s}, \bar{x})$ is stable for any $\tau < \tau_0$ and unstable for any $\tau \geq \tau_0$; this is demonstrated in Figures 5 to 8 when the delay takes the values $\tau = 1$, $\tau = 4.5$, $\tau = 5$ respectively.

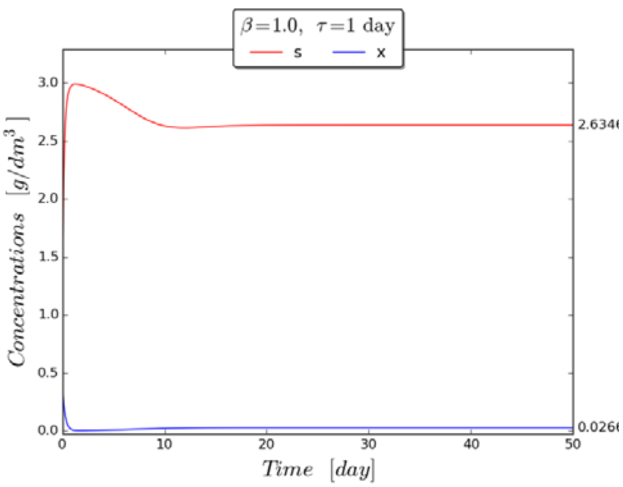

Fig. 5. $\beta = 1$, $\tau = 1$: *Time evolution of* s *and* x *towards* $\bar{s}$ *and* $\bar{x}$

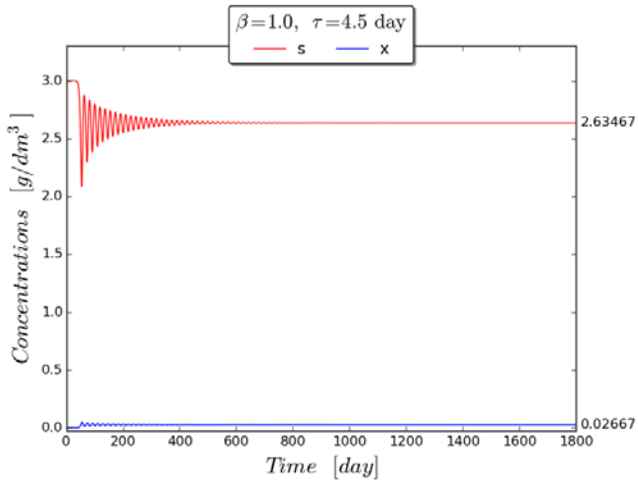

Fig. 6. $\beta = 1$, $\tau = 4.5$: *Time evolution of* s *and* x *towards* $\bar{s}$ *and* $\bar{x}$

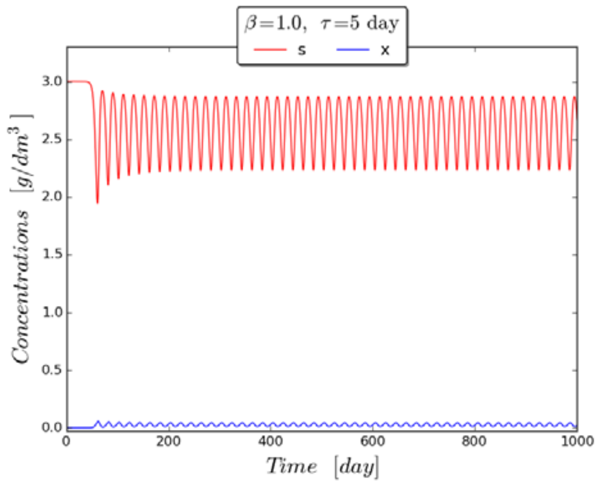

Fig. 7. $\beta = 1$, $\tau = 5$: *Time evolution of* s *and* x *towards* $\bar{s}$ *and* $\bar{x}$

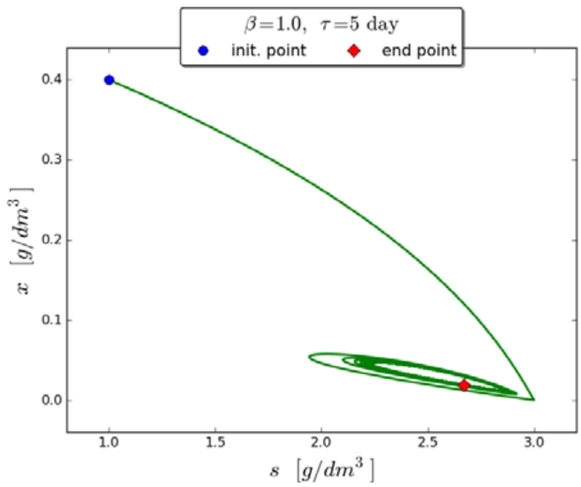

Fig. 8. $\beta = 1$, $\tau = 5$: *A trajectory in the phase plane* (s, x) *Existence of a periodic solution as a result of Hopf bifurcation*

The graphic visualizations (see Figures 2 and 5), as well as other simulations, that are not presented in the paper, show that for a delay $\tau = 1$ [day] the dynamics is stabilized approximately within 10 days, which is practically reasonable for the modeled process. Increasing the delay leads to the appearance of damped oscillations and the time for the solution stabilization increases (see Figures 3 and 6 for $\tau = 5$ and $\tau = 4.5$ respectively). For extremely large (and practically unusual) value of the delay ($\tau = 15$ [days] in Figure 4), as well as for $\tau \geq \tau_0$ (e.g. $\tau = 5$ [days] in Figures 7 and 8) sustainable oscillations do occur.

4.2. Simulation results from the extremum seeking algorithm

The numerical results from the extremum seeking algorithm (ESA) are visualized in Figures 9 to 12 for small values of the delay $\tau = 1$ and $\tau = 3$.

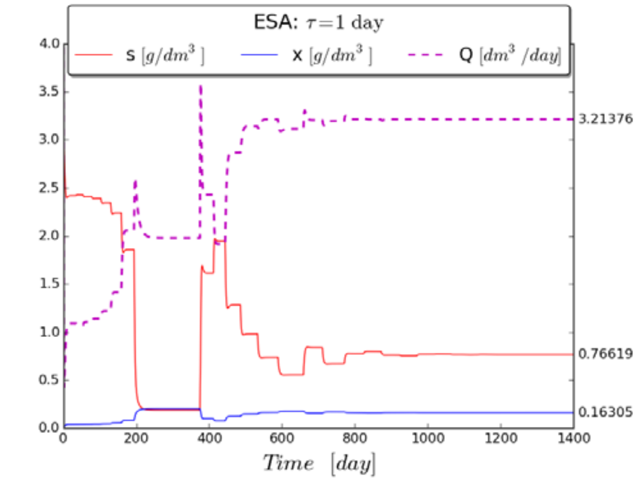

Fig. 9. $\tau = 1$. *Visualization of the numerical results from the ESA. Time evolution of* s, x *and* Q *towards* $\bar{s}_{max}$, $\bar{x}_{max}$ *and* Q_{max}

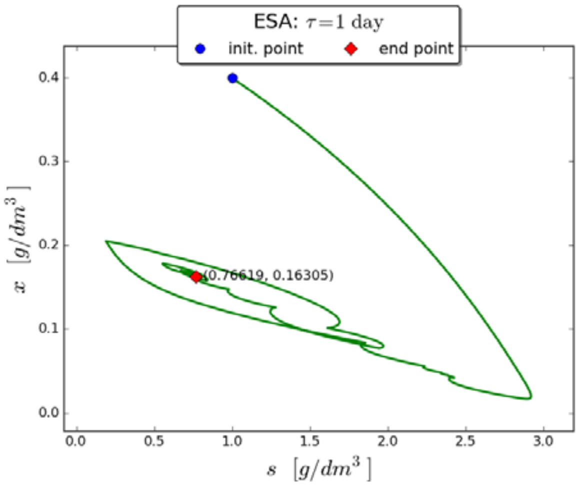

Fig. 10. $\tau = 1$. *Visualization of the numerical results from the ESA. A trajectory in the phase plane* (s, x).

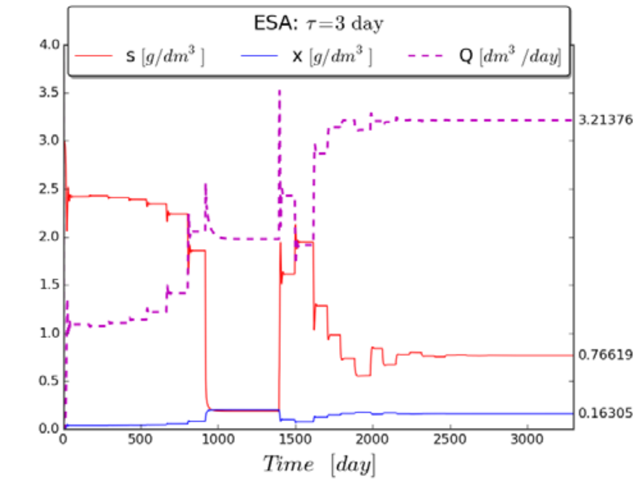

Fig. 11. $\tau = 3$. *Visualization of the numerical results from the ESA. Time evolution of* s, x *and* Q *towards* $\bar{s}_{max}$, $\bar{x}_{max}$ *and* Q_{max}

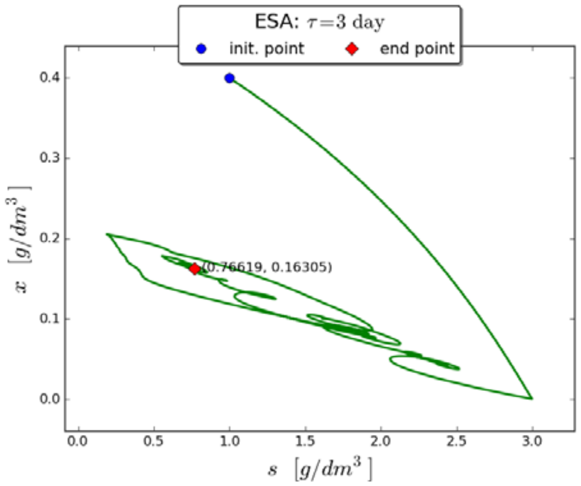

Fig. 12. $\tau = 3$. *Visualization of the numerical results from the ESA. A trajectory in the phase plane* (s, x).

5. Conclusion

In this paper we investigate a nonlinear functional-differential model of an anaerobic bioreactor with methane (biogas) production, involving one microbial population and one substrate. The stabilization of the model solutions is carried out by a feedback control low involving one discrete time delay. We have established local asymptotic stability of the system dynamics towards a previously chosen equilibrium point as well as bifurcations of this equilibrium with respect to the time delay. The numerical simulations suggest that in the cases when the equilibrium point is locally asymptotically stable it is also globally stable. This allows us to apply a numerical model-based extremum seeking algorithm for stabilizing the dynamics towards the equilibrium point where maximum production of methane is achieved. The facilities of the algorithm are illustrated numerically as well.

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