THE NONLOCAL PROBLEM FOR SECOND-ORDER DIFFERENTIAL EQUATIONS WITH THE OPERATORS OF INVOLUTION

НЕЛОКАЛЬНАЯ ЗАДАЧА ДЛЯ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ВТОРОГО ПОРЯДКА С ОПЕРАТОРАМИ ИНВОЛЮЦИИ

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Abstract: *The spectral properties of the Sturm-Liouville operator whose potential is a first-order polynomial with coefficients that contain the involution operator are studied. The boundary conditions are not strong regular for Birkhoff. It is established that the operator of the problem contains in the system of root functions an infinite number of associated functions.The spectral properties of the operator of this problem are analyzed and the conditions for the existence and uniqueness of its solution are established. It is also proved that the system of root functions of the analyzed problem forms a Riesz basis.*

KEYWORDS*: OPERATOR OF INVOLUTION, DIFFERENTIAL-OPERATOR EQUATION, EIGENFUNCTION, RIESZ BASIS, NONLOCAL PROBLEM*

1. Introduction

Suppose that H is a separable Hilbert space, $A: H \to H$ is a positive self-adjoint operator with the point spectrum $\sigma_p(A) = \{z_k \in \mathbf{R} : z_k \sim \beta k^\alpha, \alpha, \beta > 0, k = 1,2,...\}$, $V(A) \equiv \{v_k \in H : k = 1,2,...\}$ is a system of eigenfunctions that form an orthonormal basis in the space H, $H(A^s) = \{v \in H : A^s v \in H\}$, $s \geq 0$, $(u, v; H(A^s)) \equiv (A^s u, A^s v; H)$,

$$\left\| y; H(A^s) \right\|^2 \equiv \left\| A^s y; H \right\|^2,$$

$$H_1 \equiv L_2((0,1), H) = \{u(t) : (0,1) \to H, \|u(t); H\| \in L_2(0,1)\},$$

$D_x : H_1 \to H_1$ is a strong derivative in the space H_1, i.e. $\left\| \frac{u(x + \Delta x) - u(x)}{\Delta x} - D_x u; H \right\| \to 0, (\Delta x \to 0)$, $I : L_2(0,1) \to L_2(0,1)$ is the operator of involution: $Iu(x) \equiv u(1-x)$, $(u(x) \in L_2(0,1))$, $p_0 \equiv \frac{1}{2}(E + I)$, $p_1 \equiv \frac{1}{2}(E - I)$, $L_{2,j}(0,1) \equiv \{y \in L_2(0,1) : y = p_j y\}$,

$$H_{1,j} \equiv \{y(x) \in H_1 : y(x) \equiv p_j y(x)\},\ j = 0,1,$$

$$H_2 \equiv \{y(x) \in H_1 : D_x^2 y \in H_1, A^2 y \in H_1\},$$

$\|y; H_2\|^2 \equiv \left\| D_x^2 y; H \right\|^2 + \left\| A^2 y; H \right\|^2$, $L(H(A^m), H(A^q))$ is the algebra of bounded linear operators $S : H(A^m) \to H(A^q), (m, q \geq 0)$,

$$H(A^0) = H,\ L(H(A^m)) \equiv L(H(A^m), H(A^m)),$$

$H^1 \equiv H\left(A^{\frac{3}{2}}\right), H^2 \equiv H\left(A^{\frac{1}{2}}\right)$. Consider the following problem:

$$L(D_x, A)y \equiv -D_x^2 y(x) + A^2 y + 2B_0(2x-1)(y(x) + y(1-x)) = f(x), \quad (1)$$

$$l_1 y \equiv b_1 y(0) - b_2 y(1) = h_1, \qquad l_2 y \equiv D_x y(0) - D_x y(1) = h_2, \quad (2)$$

$f(x) \in H_1, h_1 \in H^1, h_2 \in H^2$.

We interpret the solution [1, 2] of the problem (1), (2) as the function $y(x) \in H_2$ satisfying the equalities

$$\|L(D_x, A)y - f; H_1\| = 0, \left\| l_1 y - h_1; H^1 \right\| = \left\| l_2 y - h_2; H^2 \right\| = 0. \quad (3)$$

The differential equation (1) includes the operator of involution. The properties of the spectral problems for second order differential equations with involution investigated in the paper [3-9].

Theorem 1.1. *Let $B_0 \in L(H_2)$. Then the operator L of the problem (1), (2), has the system of root functions $V(L)$ which forms a Riesz basis in H_1.*

Theorem 1.2. *Let $B \in L(H^1)$, $B_0 \in L(H_2)$. Then for any $f \in H_1, h_1 \in H^1, h_2 \in H^2$, there exists a unique solution of the problem (1), (2)*

$$\|y; H_2\|^2 \leq C\left(\|f; H_1\|^2 + \left\| h_1; H^1 \right\|^2 + \left\| h_2; H^2 \right\|^2 \right).$$

2. Auxiliary spectral problems

We now consider the operator $L : H_1 \to H_1$, of problem (1), (2): $Ly \equiv L(D_x, A)y$, $y \in D(L)$, $D(L) \equiv \{y \in H_2 : l_1 y = 0, l_2 y = 0\}$, Solutions of the spectral problem

$$L(D_x, A)y \equiv -D_x^2 y(x) + A^2 y + 2B_0(D_x y(x) + D_x y(1-x)) = \lambda y(x), \lambda \in C, \quad (4)$$

$$l_1 y \equiv b_1 y(0) - b_2 y(1) = 0,\ l_2 y \equiv D_x y(0) - D_x y(1) = 0 \quad (5)$$

consider as the product $y(x) = u(x) v_k$, $k = 1,2,...$ .To determine the function $u(x)$ we obtain the spectral problem

$$L_k(D_x, z_k)u \equiv -D_x^2 u(x) + z_k^2 u + 2b_{0,k}(2x-1)(u(x) + u(1-x)) = \lambda u(x), \quad (6)$$

$$l_{1,k} u \equiv b_1 u(0) - b_2 u(1) = 0,\ l_{2,k} u \equiv D_x u(0) - D_x u(1) = 0. \quad (7)$$

Consider the particular case of the problem (6), (7) if the specified conditions, $b_1 = -b_2 = 1$, $B_0 = 0$.

$$-D_x^2 u(x) + z_k^2 u = \lambda u(x), \quad (8)$$

$$u(0) - u(1) = 0,\ D_x u(0) - D_x u(1) = 0. \quad (9)$$

Lema 2.1. *Let $b_1 = -b_2 = 1$, $B_0 = 0$. Then the problem (8), (9) have point spectrum $\sigma_k \equiv \{\lambda_{k,n} \in \mathbf{R} : \lambda_{k,n} = (2\pi n)^2 + z_k^2, n = 0,1,...\}$, and the system of eigenfunctions*

$$T \equiv \left\{ \begin{array}{l} t_n^s \in L_2(0,1): \\ t_0^0(x) = 1, t_n^0(x) = \sqrt{2}\cos 2\pi n x, t_n^1(x) = \sqrt{2}\sin 2\pi n x, n \in N \end{array} \right\}.$$

We now consider the operator $L_{0,k} : L_2(0,1) \to L_2(0,1)$ of the problem (8), (7)

$$L_{0,k}u \equiv L(D_x, z_k)u\ ,\ u \in D(L_{0,k})\ ,$$
$$D(L_{0,k}) \equiv \{u \in W_2^2(0,1): l_{1,k}u = 0, l_{2,k}u = 0\}.$$

Let,

$$v_{k,0}^{0,0}(x) \equiv 1 + \beta(2x-1),\ v_{k,n}^{0,0}(x) \equiv \sqrt{2}\sin 2\pi nx\ , \quad (10)$$

$$v_{k,n}^{1,0}(x) \equiv \sqrt{2}(1+\beta(2x-1))\cos 2\pi nx\ , \quad (11)$$

$$\beta \equiv (b_1 - b_2)^{-1}(b_1 + b_2). \quad (12)$$

You can check that

$$L(D_x, z_k)v_{k,n}^{1,0}(R_0) = \lambda_{k,n}v_{k,n}^{1,0} + \xi_{k,n}^0 v_{k,n}^{0,0}\ ,$$

$$\xi_{k,n}^0 = 8\pi n\beta_{k,n}\ ,\ n = 1,2.... \quad (13)$$

Hence, $V(L_{0,k}) \equiv \{v_{k,n}^{s,0}(x); s = 0,1, n = 1,2,...\}$ is the system of root functions of the operator $L_{0,k}$ in the sense of the equality (13).

Lema 2.2. *Let $b_1 \neq b_2$. Then the operator $L_{0,k}$ of the problem (8), (7) has the point spectrum σ_k, and the system $V(L_{0,k})$ of root functions is complete and minimal in $L_2(0,1)$.*

Proof. We now prove the completeness of the system $V(L_{0,k})$ in the space $L_2(0,1)$.

Consider the adjoint problem $-D^2w(x) + {z_k}^2 w(x) = \mu w(x)$, $w(0) - w(1) = 0$, $b_2 D_x w(0) + b_1 D_x w(1) = 0$. The operator of this adjoint problem has the point spectrum σ_k, and the system of root functions

$$W(L_{0,k}) \equiv \begin{Bmatrix} w_{k,n}^{s,0} \in L_2(0,1): w_{k,q}^{0,0} = \sqrt{2}\cos 2\pi qx, \\ w_{k,q}^{1,0} = \sqrt{2}(1-\beta)\sin 2\pi qx \end{Bmatrix},\ \ (k = 1,2...).$$

Hence, the system of root functions $V(L_{0,k})$ of the operator $L_{0,k}$ possesses a unique biorthogonal system $W(L_{0,k})$

$(v_{k,n}^{r,0}, w_{q,n}^{s,0}; L_2(0,1)) = \delta_{r,s}\delta_{k,q}, (r,s = 0,1, q,k =,1...)$. Consider the operators $R_{0,k}, S_{0,k} : L_2(0,1) \to L_2(0,1)$,

$$R_{0,k}t_{k,n}^p \equiv v_{k,n}^{p,0}, R_{0,k} = E + S_{0,k}\ ,\ p = 0,1, n = 0,1,....$$

From the definition of the operator $R_{0,k}$ and the completeness of system $V(L_{0,k})$ in the space $L_2(0,1)$ we get $S_{0,k} : L_{2,1}(0,1) \to 0, S_{0,k} : L_{2,0}(0,1) \to L_{2,1}(0,1)$.

Then $S_{0,k}S_{0,k} : L_{2,0}(0,1) \to 0, S_{0,k}S_{0,k} : L_{2,1}(0,1) \to 0$, i.e., $S_{k,0}S_{k,0} = O$, where O is the zero operator in the space $L_2(0,1)$. Thus, $R_{k,0}^{-1} = E - S_{k,0}$.

To prove that the system $V(L_{0,k})$ forms a Riesz basis [3, 4] in $L_2(0,1)$, it is sufficient, according to the formula $R_{0,k} = E + S_{0,k}$, to show that the operator $S_{k,0} : L_2(0,1) \to L_2(0,1)$ is bounded.

Let ω be an arbitrary element from the space $L_2(0,1)$. We represent ω as a Fourier series in the system T.

$$\omega = \omega_0^0 t_0^0 + \sum_{m=1}^{\infty} \omega_m^0 t_m^0 + \omega_m^1 t_m^1\ ,\ \omega_m^j = (\omega, t_m^j; L_2(0,1)).$$

According to the definition of the operator $S_{0,k}$, we find

$$S_{0,k}\omega = \beta_{k,0}(2x-1)\left(\omega_0^0 v_{k,0}^0 + \sum_{m=1}^{\infty}\omega_m^0\sqrt{2}\cos 2\pi mx\right).$$

Using the ratio $\|S_{0,k}\omega; L_2(0,1)\|^2 =$

$$= \left\|\beta(2x-1)\left(\omega_0^0 v_{k,0}^0 + \sum_{m=1}^{\infty}\omega_m^0\sqrt{2}\cos 2\pi mx; L_2(0,1)\right)\right\|^2$$

we estimate it.

Hence, the operator $R_{0,k} = E + S_{0,k}$ is bounded $L_2(0,1) \to L_2(0,1)$ and $(R_{0,k}^{-1})^* = E - S_{0,k}{}^* \in L(L_2(0,1))$. So using theorem N.K. Barry (see theorem 6. 2.1[9]) we obtain the following statement.

Theorem 2.1. *Let $b_1 \neq b_2$. Then the operator $L_{0,k}$ of problem (8), (7) has the point spectrum σ_k, and the system $V(L_{0,k})$ of root functions in the sense of equality (12) which forms a Riesz basis in $L_2(0,1)$.*

Further, we introduce the operator $L_k : L_2(0,1) \to L_2(0,1)$ of the problem (6), (7).

$$L_k u \equiv L_k(D_x, z_k)u\ ,\ u \in D(L_k)\ ,$$
$$D(L_k) \equiv \{u \in W_2^2(0,1): l_{1,k}u = 0, l_{2,k}u = 0\},$$

$v_{k,n}^{0,1}(x) = \sqrt{2}\sin 2\pi nx \in D(L_k)$, $L_k v_{k,n}^0(x) = \lambda_{k,n}v_{k,n}^0(x)$, $(k = 1,2...)$.

By the direct substitution we can show that the $v_{k,n}^{0,1}(x) = \sqrt{2}\sin 2\pi nx$, is eigenfunctions of the operator L_k,

$v_{k,n}^{0,1}(x) = \sqrt{2}\sin 2\pi nx \in D(L_k)$, $L_k v_{k,n}^{0,1}(x) = \lambda_{k,n}v_{k,n}^{0,1}(x)$, $(k = 1,2...)$.

The root function of the operator L_k, defined by the relation

$$v_{k,n}^{1,1}(x) \equiv \sqrt{2}(1+\beta(2x-1))\cos 2\pi nx + \xi_{k,n}(2x-1)^2\sin 2\pi nx, \quad (13)$$

$$\xi_{k,n} = -b_0(8\pi n)^{-1}. \quad (14)$$

Hence, $V(L_k) \equiv \{v_{k,n}^{s,1}(x); s = 0,1, n = 1,2,...\}$ is the system of root functions of the operator L_k, in the sense of equality

$$L_k(D_x, z_k)v_{k,n}^{1,1}(x) = \lambda_{k,n}v_{k,n}^{1,1}(x) + \rho_{k,n}^1 v_{k,n}^{1,0}(x), \quad (15)$$

$$\rho_{k,n}^1 = \xi_{k,n}^0 + \rho_{k,n}^0\ ,\ \xi_{k,n}^0 = 8\pi n\beta. \quad (16)$$

Show that the system $V(L_k)$ of root functions of the operator L_k possesses a unique biorthogonal system $W(L_k)$.

$$R_{1,k}, S_{1,k} : L_2(0,1) \to L_2(0,1),$$

$$R_{1,k} = E + S_{1,k},\quad R_{1,k}t_{k,n}^p \equiv v_{k,n}^{p,1},\quad R_{1,k}t_{k,0}^0 \equiv v_{k,1}^{0,1}, \quad (17)$$

$p = 0,1$, $k,n \in N$.

From the formulas (12), (13) we have, that $(S_{1,k})^2 = 0$. Show that $S_{1,k} \in L(L_2(0,1))$. To have any function $\omega \in L_2(0,1)$

$$\omega = \omega_0^0 + \sqrt{2}\sum_{m=1}^{\infty}\omega_m^0\cos 2\pi mx + \omega_m^1\sin 2\pi mx,\quad S_{1,k}\omega = \omega_0^0 +$$

$$+ \sqrt{2}\sum_{n=1}^{\infty}\omega_{k,n}^{0,1}\left(\beta_{k,n}(2x-1)\cos 2\pi nx + \xi_{k,n}(2x-1)^2\sin 2\pi nx\right),$$

$$\|S_{1,k}\omega; L_2(0,1)\|^2 \le 2\left(1 + |\beta_{k,n}|^2 + |\xi_{k,n}|^2\right)\|\omega; L_2(0,1)\|^2. \quad (18)$$

So there is operator $(R_{1,k}^{-1})^* = E - (S_{1,k})^*$ such that $(R_{1,k}^{-1})^* : T \to W(L_k)$, where $W(L_k)$ is the system functions biorthogonal to $V(L_k)$.

Hence, the system of $V(L_k)$ of the operator L_k complete and minimal in forms in $L_2(0,1)$.

From the formulas (15) we have that the system $V(L_k)$ and $V(L_{0,k})$ is squarely close. So using theorem N.K. Barry (see theorem 6. 2.3[9]) we obtain the following statement.

Theorem 2.2. *Let $b_1 \neq b_2$. Then the operator L_k of the problem (6), (7) has the point spectrum σ_k, and the system $V(L_k)$ of root functions which forms a Riesz basis in $L_2(0,1)$.*

The spectral problem (1), (2). Let $b_{1,k} \neq b_{2,k}$. Then the operator L of the problem (1), (2) has the point spectrum

$$\sigma \equiv \{\lambda_{k,n} \in R : \lambda_{k,n} \equiv 4n^2\pi^2 + z_k^2, n = 0,1..., k = 1,2,...\}$$

and the system of a root functions

$$V(L) \equiv \begin{Bmatrix} v_{k,n}^s(L) \in H_1 : v_{k,n}^s(L) = v_{k,n}^{s,1}(x)v_k, \\ s = 0,1, n = 1,2,...,k,n = 0,1,... \end{Bmatrix},$$

$$v_{k,0}(L) \equiv v_k(A),\ v_{k,n}^0(L) \equiv \sqrt{2}\cos 2\pi n x v_k(A),$$

$$v_{k,n}^1(L) \equiv \sqrt{2}(1 + b(2x-1))\sin 2\pi n x v_k(A),\ n,k \in N. \tag{19}$$

The system $V(L)$ of root functions of the operator L possesses a unique biorthogonal system $W(L) \equiv \{w_{p,m}^s \in H_1 v : w_{p,m}^s \equiv w_{p,m}^{s,1} v_m, p = 0,1,...,m = 1,2,...\}$ in the sense of equality $(v_{k,m}^j, w_{p,n}^s; H_1) = \delta_{j,s}\delta_{k,p}\delta_{m,n}$.

Hence, we obtain the following statement.

Lema 2.3. *Let $b_1 \neq b_2$. Then the operator L of the problem (1), (2) has complete and minimal in H_1 system of root functions $V(L)$. Then*

$\|R_{1,k}\omega; L_2(0,1)\| \leq C\|\omega; L_2(0,1)\|$, $\|(R_{1,k}^{-1})\omega; L_2(0,1)\| \leq C\|\omega; L_2(0,1)\|$, $C > 0$.

So using theorem N.K. Barry (see theorem 6. 2.1 [10]) we obtain the following statement.

Theorem 2.3. *The operator L of the problem (1), (2), has the system of a root functions $V(L)$ which forms a Riesz basis in the H_1.*

3. Property of the problem (1), (2)

Replaced condition (2) into equivalent terms

$$l_3 y \equiv y(0) - y(1) + B(y(0) + y(1)) = h_3,$$

$$l_2 y \equiv D_x y(0) - D_x y(1) = h_2. \tag{20}$$

Here, $h_3 \equiv (b_1 - b_2)^{-1}$, $g_1 \in H^1, g_2 \in H^2$. Consider the particular case the problem (1), (20) if the specified conditions $B = 0,\ b_0 = 0.$

$$-D_x^2 y(x) + A^2 y = g(x), \tag{21}$$

$$y(0) - y(1) = g_1,\ D_x y(0) - D_x y(1) = g_2\ g_j \in H^j, j = 1,2. \tag{22}$$

Theorem 3.1. *Let $B = 0,\ b_0 = 0.$ Then for any $g \in H_1, g_1 \in H^1, g_2 \in H^2$, there exists a unique solution of the problem (21), (22).*

Proof. We seek the solution of this problem in the form $y = u + v$, there u is the solution of the problem

$$-D_x^2 u(x) + A^2 u = g(x),\ y(0) - y(1) = 0,\ D_x y(0) - D_x y(1) = 0, \tag{23}$$

and v is the solution of the problem

$$-D_x^2 v(x) + A^2 v(x) = 0,\ v(0) - v(1) = g_1, D_x v(0) - D_x v(1) = g_2, \tag{24}$$

Consider the problem (23). We expand the functions $u(x), g(x)$ in a series in the orthonormal basis in the space H_1:

$$T_1 \equiv \{t_{k,m}^s \in H_1 : t_{k,m}^s \equiv t_m^s v_k, t_m^s \in T, v_k \in V(A)\},$$

$$u = \sum_{s,k,m} u_{k,m}^s t_{k,m}^s, u_{k,m}^s = (u, t_{k,m}^s; H_1),$$

$$g = \sum_{s,k,m} g_{k,m}^s t_{k,m}^s, g_{k,m}^s = (g, t_{k,m}^s; H_1).$$

We estimate a number

$$-D_x^2 u = \sum_{s,k,m} (2\pi m)^2 \left((2\pi m)^2 + z_k^2\right)^{-1} g_{k,m}^s t_{k,m}^s,$$

$$\|D_x^2 u; H_1\| \leq \|g; H_1\|,$$

$$A^2 u = \sum_{s,k,m} z_k^2 \left((2\pi m)^2 + z_k^2\right)^{-1} g_{k,m}^s t_{k,m}^s,\ \|A^2 u; H_1\| \leq \|g; H_1\|,$$

Hence,

$$\|u; H_2\| \leq \sqrt{2}\|g; H_1\|. \tag{25}$$

Consider the problem (24). Further, we introduce the operators, $Y_j(x, A) \equiv e^{Ax} + (-1)^j e^{A(1-x)} \in L(H^1, H_2),\quad j = 0,1.$

Expression of solution of the problem

$$v(x) = Y_0(x, A)\varphi_0 + Y_1(x, A)\varphi_1. \tag{26}$$

To determine the $\varphi_0, \varphi_1 \in H^1$ we substitute expression (26) into the condition (24) and we obtain

$$\varphi_1 = \tfrac{1}{2} W_1(0, A)^{-1} g_1, \varphi_0 = \tfrac{1}{2} W_1(0, A)^{-1} A^{-1} g_2.$$

Hence,

$$v = \tfrac{1}{2} W_1(x, A) W_1(0, A)^{-1} g_1 + \tfrac{1}{2} W_0(x, A) W_1(0, A)^{-1} A^{-1} g_2,$$

$$\|v; H_2\|^2 \leq C\left(\|g_1; H^1\|^2 + \|g_2; H^2\|^2\right). \tag{27}$$

Therefore, follows from inequalities (25), (27) inequality

$$\|y; H_2\|^2 \leq C_1\left(\|g; H_1\|^2 + \|g_1; H^1\|^2 + \|g_2; H^2\|^2\right).$$

4. Results and discussion

The Sturm-Liouville operator with polynomial potentional has its application in quantum mechanics (see [10]), the theory of PT-symmetric operators (see [11]). The obtained results can be generalized for the case when the coefficients are polynomials of the order $2n$ (see [4]). The generalization of the obtained results in the case of differential equation whose order is $2n$ can be conducted according to the scheme of investigation which is used in [5].

5. Conclusion

Spectral properties of the Sturm-Liouville operator whose potential is a first order polynomial whose coefficients contain involution operator are investigated in this paper. The boundary conditions are non-strongly regular according to Birkhoff. It is established that the operator of the problem contains infinite number of joined functions in the system of its root functions.

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