MIXED VARIATIONAL PROPERTIES FOR SOME FOURTH-ORDER BEAM PROBLEMS

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Abstract: *In this paper we study eigenvalue problems for fourth-order ordinary differential equations. These boundary problems usually describe the bending vibrations of a homogeneous beam. Our aim here is to present mixed variational forms depending on a wide class of boundary conditions. In particular, we show when the symmetry in variational formulations is available. This property ensures real spectrum of the corresponding problem. The effect of the theoretical results is illustrated by some realistic examples.*

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1. Introduction

We consider the eigenvalue problem

$$y^{IV}(x) - \big(q(x)y'(x)\big)' = \lambda r(x)y(x), \quad 0 < x < l, \qquad (1)$$

subject to some boundary conditions. Here, λ is a spectral parameter; $q(x)$ and $r(x)$ are real-valued positive and continuous functions.

Problem (1) arises in the dynamical boundary-value problem describing free bending vibrations of a homogeneous beam (see, e.g. [1]).

In order to apply the various Galerkin type numerical methods, we have to present the two-point problem (1) in a weak form, when taking into account the boundary conditions. The problems of the type under consideration are appropriate to be presented in a mixed form [2,3] for purpose to be solved numerically by means of mixed numerical methods. An advantage of the mixed variational method is that, along with the unknown function (the displacement), one find an approximation also of its second derivative [2].

Let $H^m(0,l)$ be the usual Sobolev space for a positive integer m.

Then for $z(x) \in H^2[0,l]$ from (1) we easily get the system:

$$\begin{cases} -y'' = z \\ -z'' - (qy')' = \lambda r y. \end{cases} \qquad (2)$$

2. Main Result

The principal purpose is to find from (2) a mixed variational formulation according to the boundary conditions and similarly to prove under which type of boundary conditions the resulting mixed mathematical model is symmetric.

Let us introduce the test functions $y_1(x) \in H^1(0,l)$ and $z_1(x) \in H^1(0,l)$. Multiplying the first equation from (2) by z_1 and the second by y_1 and integrating on the interval $[0,l]$, we obtain:

$$\begin{cases} -\int_0^l y'' z_1\,dx = \int_0^l z z_1\,dx \\ -\int_0^l \big[z'' - (qy')'\big] y_1\,dx = \lambda \int_0^l r y y_1\,dx, \end{cases}$$

which yields

$$\begin{cases} -y' z_1\big|_0^l + \int_0^l y' z'_1\,dx = \int_0^l z z_1\,dx \\ -\big[z' - (qy')\big] y_1\big|_0^l + \int_0^l \big[z' y'_1 + q y' y'_1\big]\,dx = \lambda \int_0^l r y y_1\,dx. \end{cases} \qquad (3)$$

We denote by ξ any of the endpoints, i.e. $\xi = 0;l$. Taking into account the boundary conditions, the products $y' z_1$ and $\big[z' - (qy')\big] y_1$ should be presented at the endpoints as symmetric expressions with respect to the pairs of functions $z;y$ and $z_1;y_1$ and thus the problem under consideration would have symmetric mixed formulation.

(A) $y' z_1$ is a symmetric expression at the endpoint $\xi = 0;l$ when the boundary conditions at $\xi = 0;l$ are of the type:

- (a_1) $y'(\xi) = 0$;
- (a_2) $y''(\xi) = 0$.

As $-y'' = z$, the boundary condition of (2) is $z(\xi) = 0$. This would be an essential condition for the system (2), hence the function z_1 would also satisfy this condition for (3);

- (a_3) $k_1 y''(\xi) + y'(\xi) = 0$, with $k_1 \neq 0$.

This boundary condition is of type $y'(\xi) - k_1 z(\xi) = 0$ for the system (2).

Thus at the point $\xi = 0;l$ we have either $y'(\xi) z_1(\xi) = 0$ or $y'(\xi) z_1(\xi) = k_1 z(\xi) z_1(\xi)$, where $k_1 \neq 0$. That means:

$$-y' z_1\Big|_0^l = k_{1,0}\, z(0)\, z_1(0) - k_{1,l}\, z(l)\, z_1(l), \qquad (4)$$

where the constants $k_{1,0}$ and $k_{1,l}$ could be equal to zero.

(B) $\left[z' - (qy')\right] y_1$ represents a symmetric expression at the endpoints, if the boundary conditions at $\xi = 0;l$ are of the type:

- (b_1) $y(\xi) = 0$.

This is an essential boundary condition, consequently the function y_1 would satisfy it;

- (b_2) $y'''(\xi) - q(\xi)y'(\xi) = 0$, which for the problem (2) is transformed to:

$$z'(\xi) - q(\xi)y'(\xi) = 0;$$

- (b_3) $y'''(\xi) - q(\xi)y'(\xi) = k_2\, y(\xi) + \lambda k_3\, y(\xi)$, with $(k_2, k_3) \neq (0,0)$.

For the problem (2) this equality takes the form:

$$z'(\xi) - q(\xi)y'(\xi) = k_2\, y(\xi) + \lambda k_3\, y(\xi).$$

Hence, it is clear that if at the endpoint $\xi = 0;l$ there is a boundary condition of type (b_1)-(b_3), then either

$$\left[z'(\xi) - q(\xi)y'(\xi)\right] y_1(\xi) = 0$$

or

$$\left[z'(\xi) - q(\xi)y'(\xi)\right] y_1(\xi) = -\left(k_2 + \lambda k_3\right) y(\xi) y_1(\xi),$$

where $(k_2, k_3) \neq (0,0)$.

That means:

$$-\left[z' - (qy')\right] y_1\Big|_0^l = \left(k_{2,0} + \lambda k_{3,0}\right) y(0) y_1(0) - \left(k_{2,l} + \lambda k_{3,l}\right) y(l) y_1(l), \qquad (5)$$

and it is possible that $\left(k_{2,\xi}, k_{3,\xi}\right) = (0,0)$ for $\xi = 0;l$.

From the variational presentation (3), as well as from (4) and (5), it follows the system:

$$\left|\begin{array}{l} \displaystyle\int_0^l y' z'_1\, dx - \int_0^l z z_1\, dx + k_{1,0}\, z(0)\, z_1(0) - k_{1,l}\, z(l)\, z_1(l) = 0 \\ \displaystyle\int_0^l z' y'_1\, dx + \int_0^l q\, y' y'_1\, dx + k_{2,0}\, y(0) y_1(0) - k_{2,l}\, y(l) y_1(l) \\ \displaystyle\qquad = \lambda \left[\int_0^l r\, y\, y_1\, dx - k_{3,0}\, y(0) y_1(0) + k_{3,l}\, y(l) y_1(l)\right]. \end{array}\right. \qquad (6)$$

The considerations above allow us to formulate the following theorem:

Theorem 1. *The mixed problem (2) has symmetric variational formulation if at any endpoint the boundary conditions are of the type* $(a_i),(b_j),\ i,j = 1,2,3$.

Let us introduce the following symmetrical bilinear forms:

$$a(u,v) = \int_0^l u' v'\, dx; \quad b(u,v) = \int_0^l r\, u\, v\, dx;$$

$$a_1(u,v) = \int_0^l q\, u' v'\, dx; \quad b_1(u,v) = \int_0^l u\, v\, dx;$$

$$c_1(u,v) = k_{1,0}\, u(0)\, v(0) - k_{1,l}\, u(l)\, v(l);$$

$$c_2(u,v) = k_{2,0}\, u(0) v(0) - k_{2,l}\, u(l) v(l);$$

$$c_3(u,v) = k_{3,0}\, u(0) v(0) - k_{3,l}\, u(l) v(l).$$

From (6), it is easy to present the variational formulation in symmetric equation

$$a(y, z_1) + a(y_1, z) + a_1(y, y_1) - b_1(z, z_1) + c_1(z, z_1) + c_2(y, y_1) = \lambda \left[b(y, y_1) - c_3(y, y_1)\right]. \qquad (7)$$

Obviously, the last presentation is symmetric with respect to the couples of functions (z, y) and (z_1, y_1).

3 Application to Some Actual Problems

In this section we consider various tasks illustrating the result of Theorem 1. Some of them contain the spectral parameter in the boundary conditions (see, e.g. [4, 8-13]).

Problem 1. First we consider a mathematical model describing free bending vibrations of a homogeneous beam of constant rigidity. Both ends are fixed elastically and on these ends the servocontrol forces are in acting [6, 7, 8]:

$$y^{IV}(x) - \left(q(x)y'(x)\right)' = \lambda r(x) y(x), \quad 0 < x < 1,$$

$$y''(0) = y''(l) = 0,$$

$$y'''(0) - q(0)y'(0) - a\lambda y(0) = 0,$$

$$y'''(1) - q(1)y'(1) - c\lambda y(1) = 0.$$

Here $q(x)$ is a positive and absolutely continuous function on $[0,1]$, *a* and *c* are real constants such that $a > 0, c < 0$.

The first two boundary conditions are of type (a_2).

Comparing with the adopted notations we have:

$$l = 1; \quad r(x) = 1; \quad k_{i,\xi} = 0, i = 1,2, \xi = 0;1,$$

whereas

$$k_{3,0} = a, \; k_{3,1} = c.$$

Also $c_1(u,v)=c_2(u,v)\equiv 0$ and on account of the sign of a and c the bilinear form $c_3(u,v)=au(0)v(0)-cu(1)v(1)$ is positive.

It should be noted that the last two boundary conditions, containing the spectral parameter λ, are related to the type (b_3).

In this way, the Problem 1 assumes symmetric mixed variational formulation.

Problem 2. Let us consider the bending vibrations of a homogeneous rod, in cross-section of which the longitudinal force acts. The left end is fixed rigidly and the right one is fixed elastically and on the some endpoints the inertial mass is concentrated [6, 9]:

$$y^{IV}(x)-\left(q(x)y'(x)\right)'=\lambda r(x)y(x),\quad 0<x<1,$$
$$y(0)=y'(0)=0,$$
$$y''(1)-b_1 y'(1)-a_1\lambda y(1)=0,$$
$$y'''(1)-q(1)y'(1)-a_2\lambda y(1)=0.$$

Here $q(x)$ is a positive and absolutely continuous function in $[0,1]$, a_1, a_2 and b_1 are real constants.

For this problem we have: $l=1;\ r(x)=1$.

The boundary conditions at the left endpoint are of type (b_1) and (a_1), respectively and furthermore

$$k_{1,0}=k_{2,0}=k_{3,0}=0.$$

The last boundary condition is of type (b_3) with

$$k_{2,1}=0,\ k_{3,1}=a_2.$$

Such being the case, Problem 2 would have symmetric mixed variational formulation if the third boundary condition is of type $(a_i),i=1,2,3$. This is fulfilled in case when $a_1=0$ only, at that the condition is of type (a_2) if $b_1=0$ and of type (a_3) otherwise.

Problem 3. Consider the spectral problem [10]:

$$y^{IV}(x)-\left(q(x)y'(x)\right)'=\lambda y(x),\quad 0<x<l,$$
$$y''(0)\sin\alpha-y'(0)\cos\alpha=0,$$
$$\left(y'''(0)-q(0)y'(l)\right)\sin\beta+y(0)\cos\beta=0,$$
$$y''(l)\sin\gamma+y'(l)\cos\gamma=0,$$
$$\left(y'''(l)-q(l)y'(l)\right)\left(c\lambda+d\right)-\left(a\lambda+b\right)y(l)=0,$$

where $q(x)$ is a positive and absolutely continuous function on the interval $[0,l]$ and $\alpha,\beta,\gamma,a,b,c,d$ are real constants such that $\alpha,\beta,\gamma\in\left[0,\frac{\pi}{2}\right]$.

First, let us note that $r(x)$ is assumed to be constant (equal to 1).

Evidently the first boundary condition is of type:

(a_1) when $\alpha=0$ and thus $k_{1,0}=0$;

(a_2) when $\alpha=\frac{\pi}{2}$ and again $k_{1,0}=0$;

(a_3) when $\alpha\in\left(0,\frac{\pi}{2}\right)$ and thus $k_{1,0}=tg\,\alpha$.

Similarly, the third boundary condition is of type:

(a_1) when $\gamma=0$ and thus $k_{1,l}=0$;

(a_2) when $\gamma=\frac{\pi}{2}$ and again $k_{1,l}=0$;

(a_3) when $\gamma\in\left(0,\frac{\pi}{2}\right)$ and thus $k_{1,l}=-\tan\gamma$.

For the second boundary condition we establish that it is of type:

(b_1) when $\beta=0$, then $k_{2,0}=k_{3,0}=0$;

(b_2) when $\beta=\frac{\pi}{2}$ and again $k_{2,0}=k_{3,0}=0$;

(b_3) when $\beta\in\left(0,\frac{\pi}{2}\right)$, consequently $k_{2,0}=-\cot\beta;\ k_{3,0}=0$.

According to Theorem 1, Problem 3 assumes symmetric mixed variational formulation if the last boundary condition is of type $(b_j),j=1,2,3$. The only way of this is $c=0$, which yields that the last boundary condition is of type:

(b_3) when $a\neq 0,b\neq 0,d\neq 0$, at that $k_{2,l}=\frac{b}{d},k_{3,l}=\frac{a}{d}$;

(b_2) when $a=b=0,d\neq 0$, thus $k_{2,l}=k_{3,l}=0$;

(b_1) when $(a,b)\neq(0,0),d=0$ and $k_{2,l}=k_{3,l}=0$.

As a conclusion, Problem 3 has got a symmetric mixed variational representation in case when $c=0$ only.

Problem 4. This problem is derived from a wave equation which describes the vibration of a nonhomogeneous rod or beam clamped at one end (e.g., [5, 7, 11]):

$$y^{IV}(x)-\left(q(x)y'(x)\right)'=\lambda r(x)y(x),\quad 0<x<l,$$
$$y(0)=y'(0)=0,$$
$$y''(l)\sin\gamma+y'(l)\cos\gamma=0,$$
$$\left(y'''(l)-q(l)y'(l)\right)\sin\delta-y(l)\cos\delta=0,$$

where the coefficients *r(x)* and *q(x)* are assumed to be real-valued and continuous functions, $r(x)>0$, $\gamma\in\left[0,\frac{\pi}{2}\right]$, $\delta\in[0,\pi]$.

The boundary conditions at the left endpoint are of type (b_1) and (a_1), respectively, so that for Problem 4

$$k_{1,0}=k_{2,0}=k_{3,0}=0.$$

Just like for Problem 3, the third boundary condition is of type:

(a_1) when $\gamma = 0$ and thus $k_{1,l} = 0$;

(a_2) when $\gamma = \frac{\pi}{2}$ and again $k_{1,l} = 0$;

(a_3) when $\gamma \in \left(0, \frac{\pi}{2}\right)$ and thus $k_{1,l} = -\tan\gamma$.

In much the same manner, the fourth boundary condition is of type:

(b_1) when $\delta = 0$ or $\delta = \pi$, then $k_{2,l} = k_{3,l} = 0$;

(b_2) when $\delta = \frac{\pi}{2}$ and again $k_{2,l} = k_{3,l} = 0$;

(b_3) when $\delta \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$ and in this case $k_{2,l} = \cot\delta;$ $k_{3,l} = 0$.

Ultimately, in any case Problem 4 have symmetric mixed variational formulation.

Problem 5. At last we consider one more problem for which the spectral parameter appears in the boundary conditions [12, 13]:

$$y^{IV}(x) - \left(q(x)y'(x)\right)' = \lambda r(x)y(x), \quad 0 < x < \pi,$$
$$y(0) = y'(0) = y''(\pi) = 0,$$
$$y'''(\pi) - q(\pi)y'(\pi) + m\lambda y(\pi) = 0.$$

Here, $q(x)$ and $r(x)$ are real-valued functions from the space $C[0,\pi]$, $r(x) > 0$, and $m \in R$ is a physical parameter [13] and $l = \pi$.

The first three boundary conditions are of type (b_1), (a_1) and (a_2), respectively, so that

$$k_{1,0} = k_{2,0} = k_{3,0} = 0 \text{ and } k_{1,\pi} = 0.$$

The last boundary condition is of type (b_3), and $k_{2,\pi} = 0$, $k_{3,\pi} = -m$.

Therefore, $c_1(u,v) = 0$, $c_1(u,v) = 0$, $c_3(u,v) = m\,u(\pi)v(\pi)$ and in any case Problem 5 have symmetric mixed variational formulation.

3. Conclusions

The paper contains a useful result related to the mixed formulation of fourth-order eigenvalue problems. Namely, requirements on the boundary conditions providing symmetrical presentation of the problem are proved. This result is demonstrated for various beam problems.

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