

# APPLICATION OF LAPLACE TRANSFORM IN FINANCE

Daci A. PhD, Tola S. PhD.

Faculty of Mathematical and Physical Engineering – Polytechnic University of Tirana

alfreddaci@gmail.com, saimir\_tola@yahoo.com

**Abstract:** Laplace's transformation is an important chapter of Mathematical Analysis. At present it is widely used in various problems of signal theory, physics, mechanics, electro – techniques and economics. Laplace's transformation is the simplest and most economical method that leads directly to the required resolution of many of the various technical problems that arise when solving them. In this paper we will become acquainted with the basic concepts of operational mathematics and its application in economy. Not many analytic solutions exist for present discounted value problems but by using Laplace transform we can deduce some of the closed form solutions quite easily. There will be shown the connection between the current discounted value in finance and Laplace transformation.

**Keywords:** LAPLACE TRANSFORM, FINANCE, PRESENT VALUE, CASH FLOW, RATE, PERIOD.

## 1. Introduction

Before we consider the transformation of Laplace, we analyze the meaning of the integral of a complex function of the real argument.

Let it be  $f(t)$  a complex function of the real  $t$  argument given in the segment  $[a, b]$ . As it is known, such a function may appear in the form:  $f(t) = u(t) + iv(t)$

The integral of the function  $f(t)$  in the segment  $[a, b]$  with the definition is determined by the equation

$$\int_a^b f(t) dt = \int_a^b u(t) dt + i \int_a^b v(t) dt \quad (1)$$

Similarly, the infinite integral of that function is defined in  $(0, \infty)$ :

$$\int_0^{+\infty} f(t) dt = \int_0^{+\infty} u(t) dt + i \int_0^{+\infty} v(t) dt \quad (2)$$

provided that both right-side integrals are convergent.

Since the integral of a complex function of the real variable is defined by the equations (1) or (2) by two common integrals, then all the properties and the integration methods of the latter are true also for the integrals of the complex functions of the real argument.

Below we will be dealing with the integral of the forms (1) and (2). Considering the fact that the purpose of introducing the operational method is mainly the implementation direction of this method, we are not dwelling on the necessary conditions of convergence of such integrals but will be limited to the integral of the functions  $f(t)$  that meets these conditions: [1], [2], [3], [4]

1. The function  $f(t)$  is continuous in  $(0, +\infty)$  combined with its derivatives up to any necessary order, with the exception of possibly a finite number of first type of critical points in each finite segment.
2.  $f(t) = 0$  for  $t < 0$ .
3. The function  $f(t)$  increases in absolute value no sooner than an exponential function. So there are numbers  $M > 0$  and  $S_0 > 0$  such that for each term  $t$  we have:  $|f(t)| < Me^{S_0 t}$

The number  $S_0$  that enjoys the above quality is called the growth indicator of the function  $f(t)$ . In particular when the function  $f(t)$  is limited then it can be consider  $S_0 = 0$ . The fulfillment of the above conditions ensures the existence of many integrals of the

form (2) that we will consider below. We are now going to determine the transformation of Laplace.

Consider the product of the function  $f(t)$  with the real argument  $t$

$$e^{-pt} \cdot f(t) \quad (3)$$

Function (3) is again a complex function of the real argument:

$$e^{-pt} \cdot f(t) = e^{-(s+i\omega)t} \cdot f(t) = e^{-st} \cdot f(t) \cdot e^{-i\omega t} = e^{-st} f(t) \cos \omega t - i e^{-st} f(t) \sin \omega t$$

We will now show that when a function  $f(t)$  meets the above conditions, then the integral

$$\int_0^{+\infty} e^{-pt} f(t) dt = \int_0^{+\infty} e^{-st} f(t) \cos \omega t dt - i \int_0^{+\infty} e^{-st} f(t) \sin \omega t dt \quad (4)$$

converges, even absolutely provided that:  $\text{Re } p = s > S_0$ .

According to the definition for the convergence of the integral (4), the existence of each of the right-side integrals is necessary and sufficient. Let's analyze the convergence of the first.

$$\left| \int_0^{+\infty} e^{-st} f(t) \cos \omega t dt \right| < \int_0^{+\infty} |e^{-st} f(t) \cos \omega t| dt < < M \int_0^{+\infty} e^{-st} e^{S_0 t} dt = M \int_0^{+\infty} e^{-(s-S_0)t} dt = \frac{M}{s - S_0}$$

Similarly, the second integral on the right side of (4) is evaluated.

The integral (4) converges to the entire complex plot area  $p$  extending to the right of the line  $\text{Re } p = s_0$ . The proving method also follows that only the real part of the number  $p$  determines the convergence of the integral (4).

Integral (4) defines a function of the parameter we mark  $F(p)$ , namely:

$$F(p) = \int_0^{+\infty} f(t) e^{-pt} dt.$$

The function  $F(p)$ , is called the Laplace image of function  $f(t)$ . The function  $f(t)$  is called the original function.

As has been stated above, it follows that the fulfillment of the above three conditions is sufficient for the function  $f(t)$  to be original. Below we will call original only the functions that meet those three conditions. This tightening of the class of originals is not very sensible from the practical point of view.

If  $F(p)$ , is a reflection of the function  $f(t)$ , then it says:

$$L\{f(t)\} = F(p) \text{ or } F(p) \xrightarrow{\square} f(t)$$

**Theorem.** If the function  $F(p)$ , is an image, then:

$$F(p) \rightarrow 0 \text{ when } \operatorname{Re} p = s \rightarrow +\infty$$

Truly, it is known that  $|a + ib| = \sqrt{a^2 + b^2} \leq |a| + |b|$

$$\text{So: } |F(p)| \leq \left| \int_0^{+\infty} e^{-st} f(t) \cos \omega t dt \right| + \left| \int_0^{+\infty} e^{-st} f(t) \sin \omega t dt \right|$$

or

$$|F(p)| \leq \frac{M}{s - s_0} + \frac{M}{s - s_0} \rightarrow 0 \text{ for } s \rightarrow +\infty$$

Based on this theorem it can be said that the necessary condition for a given function  $F(p)$ , to be an image of a function  $f(t)$  is to complete the reconciliation:

$$\lim_{\operatorname{Re} p \rightarrow +\infty} F(p) = 0$$

## 2. Finding the original when the image is rational

In the practice of the usage of the operational calculus, we often find it when the function  $F(p)$ , is rational. By the end of this paragraph we will deal precisely with the problem of finding the original when the image is rational:

$$F(p) = \frac{b_0 p^m + b_1 p^{m-1} + \dots + b_{m-1} p + b_m}{a_0 p^n + a_1 p^{n-1} + \dots + a_{n-1} p + a_n} = \frac{\phi(p)}{\psi(p)} \quad (5)$$

We note, first, that the rational fraction  $F(p)$  should be regular because otherwise the condition is not met:

$$F(p) \xrightarrow{|p| \rightarrow \infty} 0$$

which is necessary for the function  $F(p)$ , to be an image. So we will have  $m < n$ .

In relation to the denominator  $\psi(p)$  of the fraction (5) we distinguish these cases:

**I. The function  $\psi(p)$  has only simple roots (not repeated).**

We mark the roots of the polynomial  $\psi(p)$  with  $\alpha_1, \alpha_2, \dots, \alpha_n$ . Then it can be written in the form:

$$\psi(p) = (p - \alpha_1)(p - \alpha_2) \dots (p - \alpha_n)$$

while the function  $F(p)$ , can be expanded:

$$\frac{\phi(p)}{\psi(p)} = \frac{A_1}{p - \alpha_1} + \frac{A_2}{p - \alpha_2} + \dots + \frac{A_i}{p - \alpha_i} + \dots + \frac{A_n}{p - \alpha_n} \quad (6)$$

Since for the function  $\frac{A_i}{p - \alpha_i}$  the original is  $A_i e^{\alpha_i t}$ , then

based on the property of linearity and equation (6) we will have:

$$\frac{\phi(p)}{\psi(p)} \xrightarrow{\bullet} A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} + \dots + A_n e^{\alpha_n t} \quad (7)$$

To find the coefficients we: multiply the two sides of (6) with  $p - \alpha_1$  and using the fact that  $\alpha_1$  is the root of  $\psi(p)$  i.e  $\psi(\alpha_1) = 0$ , the draw (6) takes the form:

$$(p - \alpha_1) \frac{\phi(p)}{\psi(p) - \psi(\alpha_1)} = A_1 + (p - \alpha_1) \left[ \frac{A_2}{p - \alpha_2} + \dots + \frac{A_n}{p - \alpha_n} \right]$$

Now we get the limit on both sides of this draw for  $p \rightarrow \alpha_1$ . It is clear that the right-side borders will be  $A_1$ . So we will have:

$$A_1 = \lim_{p \rightarrow \alpha_1} (p - \alpha_1) \frac{\phi(p)}{\psi(p) - \psi(\alpha_1)} = \lim_{p \rightarrow \alpha_1} \frac{\phi(p)}{\frac{\psi(p) - \psi(\alpha_1)}{p - \alpha_1}}$$

$$\text{from where } A_1 = \frac{\phi(\alpha_1)}{\psi'(\alpha_1)}$$

In the role of  $\alpha_1$  it could be each of the roots  $\alpha_2, \dots, \alpha_i, \dots, \alpha_n$ , therefore in the general form we will have:

$$A_i = \frac{\phi(\alpha_i)}{\psi'(\alpha_i)} \quad (8)$$

Based on (7) and (8) we can write the operational relation:

$$\frac{\phi(p)}{\psi(p)} \xrightarrow{\bullet} \frac{\phi(\alpha_1)}{\psi'(\alpha_1)} e^{\alpha_1 t} + \frac{\phi(\alpha_2)}{\psi'(\alpha_2)} e^{\alpha_2 t} + \dots + \frac{\phi(\alpha_n)}{\psi'(\alpha_n)} e^{\alpha_n t}$$

or

$$\frac{\phi(p)}{\psi(p)} \xrightarrow{\bullet} \sum_{i=1}^n \frac{\phi(\alpha_i)}{\psi'(\alpha_i)} \cdot e^{\alpha_i t} \quad (9)$$

**II. Among the roots of the denominator  $\psi(p)$  there are such that are repeated.**

Let  $\alpha, \beta, \dots, \lambda$  be the roots of the polynomial  $\psi(p)$ , while the numbers  $a, b, \dots, l$  indicate how many times each of them is rooted. In this case, the polynomial  $\psi(p)$  may appear in the form:

$$\psi(p) = (p - \alpha)^a (p - \beta)^b \dots (p - \lambda)^l$$

As it is known from the expansion of regular rational fractions into elemental fractions, each factor of the form  $(p - \delta)^d$  corresponds  $d$  to the elemental fraction with denominator

$$(p - \delta)^d (p - \delta)^{d-1} \dots (p - \delta).$$

So the function  $F(p) = \frac{\phi(p)}{\psi(p)}$  in this case will be presented in the form:



At the end of the second period:

Interest matured  $[P \cdot (1+r)] \cdot r$

The accumulated value

$$P(1+r) + [P \cdot (1+r)] \cdot r = P(1+r)^2$$

At the end of the third period:

Interest matured  $[P \cdot (1+r)^2] \cdot r$

The accumulated value

$$P(1+r)^2 + [P \cdot (1+r)^2] \cdot r = P(1+r)^3$$

Continuing this way we would get the accumulated values:

$$P(1+r), P(1+r)^2, P(1+r)^3, \dots$$

They form a geometric progression  $t$  whose term is:

$$S = P(1+r)^t \quad (15)$$

#### 4. Relation Between Present Value and Laplace Transform

In business transactions it is usually necessary to determine the level of principal  $P$  today, that will be accumulated at a compound interest rate  $r$  in a given sum  $S$  on a given date ( $t$  interest period from that moment). By formula (15)

we have:

$$P = \sum_{t=1}^T \frac{S(t)}{(1+r)^t} \quad (16)$$

In other words, it is the amount that we would be willing to pay today in order to receive a cash flow or a series of them in the future. [6] Now by using an exponential series we can write equation (16) as,

$$P = \sum_{t=1}^T e^{-rt} S(t) \quad (17)$$

In the limiting case replacing summation to an integral, equation (17) can be written as

$$P = \int_0^T e^{-rt} S(t) dt \quad (18)$$

Again here  $T$  is some finite quantity. So if we consider as  $T \rightarrow +\infty$ , equation (18) will becomes

$$P = \int_0^{+\infty} e^{-rt} S(t) dt \quad (19)$$

Equation (19) is nothing other than the definition of Laplace's transformation, hence: [6]

$$P(r) = L\{S(t)\} \quad (20)$$

Suppose successive  $t$  payments are to be made,  $k_1, k_2, k_3, \dots, k_t$

where:  $k_1$  to be settled after a year,  $k_2$  after two years and so on.

How much is the amount to be deposited today in one bank account in order to get enough savings to afford all future payments when the annual interest rate is  $r$ .

So what is the present value of all payments?

We start from formula (15) and apply it for the foregoing. To have the quantity  $k_1$  after one year we have:  $k_1 = P(1+r)$ , from where,

the amount we need to deposit today is  $P = \frac{k_1}{1+r}$ . To have the

quantity  $k_2$  after two years;  $k_2 = P(1+r)^2 \Rightarrow P = \frac{k_2}{(1+r)^2}$ , and

so on. The full amount that needs to be deposited today, so that all payments are covered is:

$$P_t = \frac{k_1}{1+r} + \frac{k_2}{(1+r)^2} + \frac{k_3}{(1+r)^3} + \dots + \frac{k_t}{(1+r)^t} \quad (21)$$

If the payments are the same,  $k_1 = k_2 = k_3 = \dots = k_t = k$  then

$$P_t = \frac{k}{1+r} + \frac{k}{(1+r)^2} + \frac{k}{(1+r)^3} + \dots + \frac{k}{(1+r)^t} \quad (22)$$

This is nothing but a series of geometric progression at a time  $\frac{1}{1+r}$  that;

$$P_t = \frac{k}{1+r} \cdot \frac{1 - \frac{1}{(1+r)^t}}{1 - \frac{1}{1+r}} = \frac{k}{r} \left[ 1 - \frac{1}{(1+r)^t} \right] \quad (23)$$

If we use Laplace transformation

$$L\{k\} = k \int_0^{+\infty} e^{-rt} dt = \frac{k}{r} \quad (24)$$

Example. Estimate the current value of a \$ 1000 instant spill series at the end of each year when the annual interest rate is 10%.

Solution:  $P = \frac{k}{r} = \frac{1000}{10\%} = 10000\$$

#### 4. Conclusions

- 1) Laplace transformation is the simplest and most economical method that leads directly to the required resolution of many of the various technical problems that arise when solving them.
- 2) Using the Laplace transformation we easily obtain a definitive analytical solution to the problems of the present discounted value where was shown the close relationship existing between the actual discounted value in finance and the transformation of Laplace. [6]
- 3) The result in this paper increases the practical benefits of Laplace transformation particularly in finance.
- 4) Laplace transformation is the major resource for discounted value functions to illustrate complex problems.

#### 5. References

1. Jorgo Malita, *Analiza Komplekse*, Tiranë 2009.
2. Joel L. Schiff, *The Laplace Transform, Theory and Applications*, Springer 1999.
3. Marcel B. Finan, *Laplace Transforms: Theory, Problems, and Solutions*, Arkansas Tech University.
4. C.T.J. Dodson, *Introduction to Laplace Transforms for Engineers*, School of Mathematics, Manchester University.
5. Omer Stringa, *Matematika Financiare*, Pegi, Tiranë 2004.
6. N. A. Patil, Vijaya N. Patil, *Application of Laplace Transform*, Global Journals Inc. (USA), Online ISSN: 2249-4626 & Print ISSN: 0975-5896, Volume 12 Issue 12 Version 1.0 Year 2012.