INVESTIGATION OF CONVERGENCE OF ξ APPROXIMATIONS ON COMPLEX NUMBER PLANE

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Abstract: **:** *In this study, the convergence behaviour of the ξ approximants for the exponantial operators is investigated on the entire complex plain. The purpose of this study is to develop an algorithm to observe how to transform a initial region on the complex plane defined by ξ approximants.*

Keywords: LIE OPERATORS, COMPLEX RECURSIVE FUNCTIONS, EVALUATION OPERATORS

1. Introduction

In this study, the convergence behavior of the ξ approximants for the exponential operators is investigated on the entire complex plain.

Exponential operators play important roles in many branches of science and engineering. They can be constructed to characterize the solutions of various mathematically formulated problems. Exponential operators give quite accurate results for the short-term evolution but causes error accumulation for long-term evolution when they are employed through a numerical approximation scheme based on time discretization. Hence we need to evaluate exponential operators in a global and rapidly converging manner.

System with n degrees of freedom will be characterized by x_1, $x_2, \dots, x_n$ complex variables, which are considered as the coordinates of a point or components of a vector in an n-dimensional complex vector space.

$\{ x_1(t), x_2(t), \dots, x_n(t)\}$: System

$x_1,\ x_2, \dots, x_n$: Phase space vector components in this system.

Hence $Q(t_f, t_i)$ global evolution operator is defined as

$$x(t_f) = Q(t_f, t_i).x(t_i)$$

where t_i and t_f denote the initial and final states respectively. If factorization of evolution operators is considered as a sequence of rather simple global evolution operators then the evolution operator can be written as follows

$$Q(t_f, t_i) = e^{(t_f - t_i).S}$$

where S is defined through

$$S = \sum_{j=1}^{N} f_j\ (x_1,\ x_2, \dots, x_N)\frac{\partial}{\partial x_j}$$

so Q evolution operator can be assumed to be written as

$$Q = e^{tf(x)\frac{\partial}{\partial x}} = \prod_{j=1}^{\infty} e^{\sigma_j(t)x^j\frac{\partial}{\partial x}} \qquad \sigma_j(t) = tf_j$$

where $L = f(x)\frac{\partial}{\partial x}$ and

$$f(x) = \sum_{j=1}^{\infty} f_j\, x^j \qquad |x| < \rho$$

This equation is factorization formula for the one-dimensional case.

The definition of the evolution operators and their relations with the solutions of a differential equations are presented. Also the effects of the evolution operators on the functions is obtained as follows:

for $j = 0$

$$Q^{(0)}f(x) = e^{\sigma_0(t)\frac{\partial}{\partial x}} = f\big(x + \sigma_0(t)\big)$$

for $j = 1$

$$Q^{(1)}f(x) = e^{\sigma_1(t)x\frac{\partial}{\partial x}} = f\big(e^{\sigma_1(t)}x\big)$$

for $j = n$ by $x = y^{\frac{1}{1-n}}$ transformation

$$Q^{(n)}f(x) = e^{\sigma_n(t)x^n\frac{\partial}{\partial x}} = f\left(\frac{x}{[1-(j-1)x^{j-1}\sigma_j(t)]^{\frac{1}{j-1}}}\right) \qquad j \geq 2$$

The essential approximation is to truncate

$$Q = \prod_{j=1}^{\infty} e^{\sigma_j(t)x^j\frac{\partial}{\partial x}} \qquad \sigma_j(t) = t.f_j$$

to a finite order. By this way it produces the following approximation.

$$\overline{\xi_n}(x,t) = \left\{\prod_{j=1}^{n} Q^{(j)}\right\} x$$

If the infinite product representation of Q converges then the following result can be obtained:

$$\bar{\xi}(x,t) = Qx = e^{tf(x)\frac{\partial}{\partial x}}\, x = \lim_{n\to\infty} \overline{\xi_n}$$

A recursion relation for these approximants can be shown as follows:

$$\overline{\xi_{n+1}} = \left\{\prod_{j=1}^{n} Q^{(j)}\right\} e^{\sigma_{n+1}(t)x^{n+1}\frac{\partial}{\partial x}}\ x$$

It can be clearly seen that this equation becomes

$$\overline{\xi_{n+1}} = \left\{\prod_{j=1}^{n} Q^{(j)}\right\} \frac{x}{[1 - n\sigma_{n+1}(t)x^n]^{\frac{1}{n}}}$$

As a result we can easily obtain that

$$\overline{\xi_{n+1}} = \frac{\overline{\xi_n}(x,t)}{\left[1 - n\sigma_{n+1}(t)\,\overline{\xi_n}^{\,n}(x,t)\right]^{\frac{1}{n}}}$$

And this is a recursion relation with an initial member evaluated as follows:

$$\bar{\xi_1}(x,t) = e^{\sigma_1(t)x\frac{\partial}{\partial x}}\ x = xe^{\sigma_1(t)} = xe^{f_1 t}$$

Although this is a simple recursion relation, the existence of f_1 may not be suitable for numerical purpose depending on the value of f_1. So we can normalize ξ- approximants as follows:

$$\overline{\sigma_{n+1}} = n\sigma_{n+1}e^{nf_1 t}$$

Then the final recursion relation becomes as follows:

$$\xi_{n+1}(x,t) = \frac{\xi_n(x,t)}{\left(1 - \overline{\sigma_{n+1}}(t)\xi_n^n(x,t)\right)^{\frac{1}{n}}}$$

The relation between the final and the previous approximants can be given as

$$\overline{\xi_n}(x,t) = \xi_n(x,t)\, xe^{f_1 t}$$

In this study the convergence of the ξ-approximant sequences in the complex plane is main issue. The above transformation of ξ_n to ξ_{n+1} can be interpreted as applying some basic elementary transformations consecutively. For this reason the following transformations are presented

$w_1 = f_1(z) = z^n$	$0 \leq r < \infty$	$0 \leq \varphi \leq \frac{2\pi}{n}$
$w_2 = f_2(z) = -\sigma_{n+1}w_1$	$0 \leq r < \infty$	$0 \leq \varphi \leq 2\pi$
$w_3 = f_3(z) = 1 + w_2$	$0 \leq r < \infty$	$0 \leq \varphi \leq 2\pi$
$w_4 = f_4(z) = \frac{1}{w_3}$	$0 \leq r < \infty$	$-\emptyset \leq \varphi \leq \emptyset$
$w_5 = f_5(z) = w_4 - 1$	$0 \leq r < \infty$	$0 \leq \varphi \leq 2\pi$

$$w_6 = f_6(z) = \frac{1}{\sigma_{n+1}} w_5 \qquad 0 \le r < \infty \quad 0 \le \varphi \le 2\pi$$

$$w_7 = f_7(z) = w_6^{\frac{1}{n}} \qquad 0 \le r < R \quad 0 \le \varphi \le 2\pi$$

As can be easily seen we get the main transformation function when we apply these consecutive transformations.

$$w_7 = w_6^{\frac{1}{n}} = \left(\frac{1}{\sigma_{n+1}} w_5\right)^{\frac{1}{n}} = \left(\frac{1}{\sigma_{n+1}}(w_4 - 1)\right)^{\frac{1}{n}} = \left(\frac{1}{\sigma_{n+1}}\left(\frac{1}{w_3} - 1\right)\right)^{\frac{1}{n}}$$

$$= \left(\frac{1}{\sigma_{n+1}}\left(\frac{1}{1 + w_2} - 1\right)\right)^{\frac{1}{n}} = \left(\frac{1}{\sigma_{n+1}}\left(\frac{1}{1 - \sigma_{n+1} w_1} - 1\right)\right)^{\frac{1}{n}}$$

$$= \left(\frac{1}{\sigma_{n+1}}\left(\frac{1}{1 - \sigma_{n+1}\xi_n^n} - 1\right)\right)^{\frac{1}{n}} \qquad z \to \xi_n$$

$$= \left(\frac{1}{\sigma_{n+1}}\left(\frac{1 - 1 - \sigma_{n+1}\xi_n^n}{1 - \sigma_{n+1}\xi_n^n}\right)\right)^{\frac{1}{n}} = \left(\frac{\xi_n^n}{1 - \sigma_{n+1}\xi_n^n}\right)^{\frac{1}{n}} = \frac{\xi_n}{(1 - \sigma_{n+1}\xi_n^n)^{\frac{1}{n}}}$$

Here $\sigma_n(t)$ functions are assumed to be given and the ξ-approximants' nature are determined by these $\sigma_n(t)$ functions. The main purpose of the examination of this convergence is to give important information about the convergence of finite product sequences that appears during the factorization of these sequences.

The purpose of this study is to develop an algorithm to find out or observe how the above recursion relation defined via ξ-approximants transform a given initial region on the complex plane defined by ξ-approximants and a develop a computer program based on this algorithm. By this computer program we can expect to be able to examine the regional changes during the consecutive recursion transformations ant to put forward some theorems by investigating this program outputs.

A singular point taken in the domain can carry us to infinity on the ξ_{n+1}–plane as can be noticed through

$$\xi_{n+1} = \frac{\xi_n}{(1 - \sigma_{n+1}\xi_n^n)^{\frac{1}{n}}}$$

Therefore one can obtain a singularity free initial region on ξ_n –plane by determining the locations of these singularities and discarding them from the domain.

In this study, the original contribution is the separation of the recursion relation between two consecutive ξ-approximants into basic simple consecutive transformations.

The other contributions are the construction of an algorithm that evaluates the region variations through the consecutive transformations and to develop a computer program to execute this algorithm. The computer program is written in C++ language and Mathematica is used for graphics.

2. Conclusion

In this study has shown for the $\sigma_n(t)$ sequences that shrink fast enough that the equivalent in the ξ_n-plane of a circular region without a singularity for the ξ_1-plane will remain without singularity in the ξ_n series, regardless of n and time.

3. References

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