SIMULTANEOUS RESONANCE CASES IN A PITCH – ROLL SHIP MODEL. PART 2: NUMERICAL ANALYSIS

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Abstract: *In a companion paper, the response of a two-degrees-of-freedom ship model with nonlinear coupled pitch and roll modes under sinusoidal harmonic excitation was studied analytically by means of the Multiple Scales method for the case where the pitch frequency is twice the roll frequency. Five resonant cases were analysed and the governing equations for the transition towards the steady-state solutions, the first-order approximations for these solutions and the frequency-amplitude relationships were derived. The present contribution aimed to verify the accuracy of the analytical results by contrasting them with the numerical results provided by direct integration of the equations of motion. The two sets of results were found to be in excellent or, at least, in decent agreement every time the system parameters were selected without a flagrant violation of the order's magnitude.*

Keywords: SHIP ROLLING AND PITCHING, NUMERICAL ANALYSIS, INTERNAL AND EXTERNAL RESONANCES

1. Introduction

Although considerable experimental and theoretical research has been realized, the development of a full satisfactory mathematical model for estimation the ship motion in waves remains an open problem. Models with one to six degrees of freedom have been developed and studied [1 – 4].

In the present paper, the model proposed by Pan and Davies for the nonlinear coupled pitch and roll modes under harmonic excitation is considered [5]. The equations of motion, which are weakly nonlinear and coupled by quadratic terms, are as follows

$$\begin{aligned} &\ddot{x}_1 + 2\varepsilon\mu_1\dot{x}_1 + \omega_1^2 x_1 + \varepsilon\alpha_1 x_1 x_2 = F_1\cos\Omega t \\ &\ddot{x}_2 + 2\varepsilon\mu_2\dot{x}_2 + \omega_2^2 x_2 + \varepsilon\alpha_2 x_1^2 = F_2\cos\Omega t \end{aligned} \tag{1}$$

where x_1 and x_2 are the roll and pitch modal amplitudes, μ_1 and μ_2 the modal damping coefficients, ω_1 and ω_2 the natural angular frequencies, Ω the excitation (wave) frequency, F_1 and F_2 the excitation force amplitudes, α_1 and α_2 the coefficients of nonlinear terms, and ε a small parameter. The dots, as always, stand for the differentiation with respect to time *t* and all the coefficients in (1) depend on fluid parameters, ship's characteristics, etc. Deleanu has considered a straightforward expansion for the solution of system (1) and has obtained five resonant values of external excitation frequency and one internal resonance [6]. Kamel has applied the multiple scales method for finding the approximate response of system (1) in the special case of internal resonance $\omega_2 \approx 2\omega_1$ associated with the combined external resonance $\omega_1 + \omega_2 = \Omega$ [7]. In a companion paper, Deleanu has derived or just presented the governing equations for the transition towards the steady-state solutions, the first-order approximations for these solutions and the frequency-amplitude relationships for all the resonant cases, namely $\Omega \approx 0$, $\Omega \approx \omega_1/2$, $\Omega \approx \omega_1$, $\Omega \approx \omega_2$, and $\Omega \approx \omega_1 + \omega_2$ [8].

The goal of this paper is to check the accuracy of the analytical approximations derived in [7] and [8] by comparing them with the numerical solutions provided by direct integration of the equations of motion.

2. Analytical versus numerical results

In this section, we treat separately the five mentioned situations beginning with the non-resonant case, and continuing with the external resonant ones. For a better understanding of the topic, we include in each analyzed case the first-order approximate solution describing the long term behavior of system (1), as well as the governing differential equations for the transition period (see [8]).

2.1. The case Ω far from $0, \omega_1/2$, ω_1, ω_2 and $\omega_1 + \omega_2$

The first – order approximations for the solution of (1) in the transition period are written as

$$\begin{aligned} &x_1 = a_1\cos(\omega_1 t + \varphi_1) + \Lambda_1\cos\Omega t \\ &x_2 = a_2\cos(2\omega_1 t + \varphi_2 + 2\varphi_1) + \Lambda_2\cos\Omega t \end{aligned} \tag{2}$$

where the amplitudes $a_n, n = 1,2$, and phases $\varphi_n, n = 1,2$, are obtained from

$$\begin{aligned} &\dot{a}_1 = -\varepsilon\mu_1 a_1 - \frac{\varepsilon\alpha_1 a_1 a_2}{4\omega_1}\sin\varphi_2,\ a_1\dot{\varphi}_1 = \frac{\varepsilon\alpha_1 a_1 a_2}{4\omega_1}\cos\varphi_2 \\ &\dot{a}_2 = -\varepsilon\mu_2 a_2 + \frac{\varepsilon\alpha_2 a_1^2}{4\omega_2}\sin\varphi_2 \\ &a_2\dot{\varphi}_2 = a_2(\omega_2 - 2\omega_1) + \left(\frac{\varepsilon\alpha_2 a_1^2}{4\omega_2} - \frac{\varepsilon\alpha_1 a_2^2}{2\omega_1}\right)\cos\varphi_2 \end{aligned} \tag{3}$$

and $\Lambda_n = F_n/(\omega_n^2 - \Omega^2)$, $n = 1,2$.

The steady – states solutions are characterized by $a_1 = a_2 = 0$. Fig. 1 illustrates a typical time series for the x_1 component, both for transient and stationary stages. Throughout the paper, the continuous lines are used for numerical solution and the stars for analytical results provided by multiple scales method.

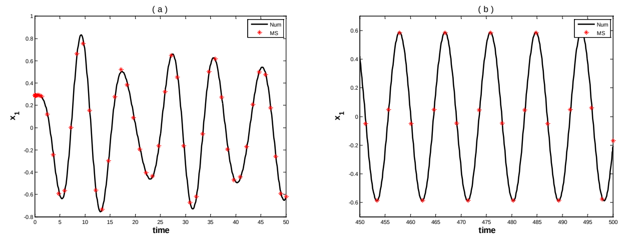

Fig. 1 *Time history for component* x_1 *and parameters values* $\varepsilon = 0.1$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\mu_1 = 0.25$, $\mu_2 = 0.5$, $\omega_1 = 1$, $\omega_2 = 2.03$, $\Omega = 0.7$, $F_1 = 0.3$ *and* $F_2 = 0.1$ *: a) transition period; b) steady – state solution.*

Using roll and pitch amplitudes given by $|\Lambda_n|$ or by numerical integration of (1), after the transients die out, one can construct the frequency - response curves shown in fig. 2. The important fixed parameters $\omega_1 = 1$ and $\omega_2 = 2.03$ were utilized. The external frequency Ω was selected in the ranges $[0.1, 0.5]$, $[0.55, 0.95]$, $[1.05, 1.95]$ and $[2.05, 2.95]$, meaning that it is relatively far from external resonances. It is obvious from fig. 2 that the approximate solution (2) with $a_1 = a_2 = 0$ matches extremely well or, at least, pretty well with its numerical counterpart. Moreover, it is worth noting that this case represents more than 90% of all possible Ω values.

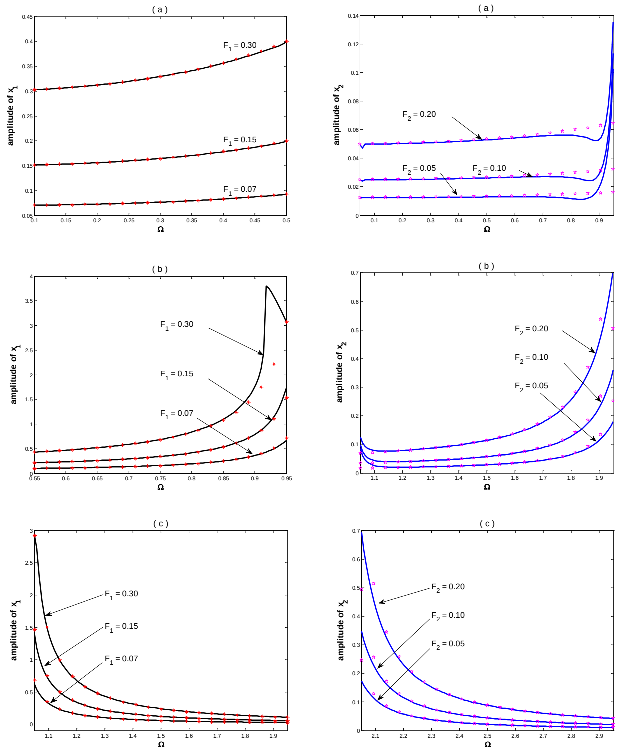

Fig. 2 *Roll and pitch amplitudes as a function of external frequency for constant parameter values* $\varepsilon=0.1$, $\alpha_1=1$, $\alpha_2=2$, $\mu_1=0.25$, $\mu_2=0.5$, $\omega_1=1$, $\omega_2=2.03$ *and different forcing amplitudes .*
Left*: Roll amplitudes for* $F_1\in\{0.07,0.15,0.3\}$ *and* $F_2=0.05$ *;*
Right: *Pitch amplitudes for* $F_1=0.07$ *and* $F_2\in\{0.05,0.1,0.2\}$.

Some notable differences appear for large forcing and only in the nearness of external resonances (see fig. 3).

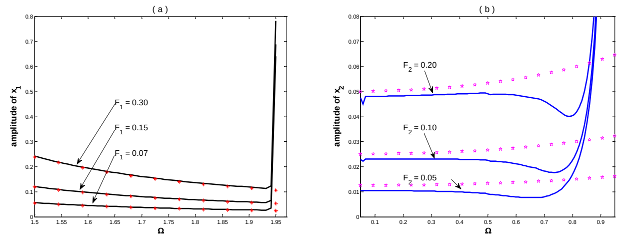

Fig. 3 *Roll and pitch amplitudes as a function of external frequency for constant parameter values* $\varepsilon=0.1$, $\alpha_1=1$, $\alpha_2=2$, $\mu_1=0.25$, $\mu_2=0.5$, $\omega_1=1$, $\omega_2=2.03$ *and different forcing amplitudes .*
a) $F_1\in\{0.07,0.15,0.3\}$ *and* $F_2=0.4$ *;*
b) $F_1=0.2$ *and* $F_2\in\{0.05,0.1,0.2\}$.

2.2. The case $\Omega\approx 0$

The long-term behavior is described by the laws

$$x_1=\frac{F_1}{\omega_1^2}\cos\Omega t,\ x_2=\frac{F_2}{\omega_2^2}\cos\Omega t \qquad (3)$$

while the system describing the motion in the transient stage is written as

$$\dot{a}_1=-\varepsilon\mu_1 a_1-\frac{\varepsilon\alpha_1 a_1 a_2}{4\omega_1}\sin\varphi_2$$

$$a_1\dot{\varphi}_1=\frac{\varepsilon\alpha_1 a_1 a_2}{4\omega_1}\cos\varphi_2+\frac{\varepsilon\alpha_1 a_1 F_2}{2\omega_1\omega_2^2}\cos\Omega t \qquad (4)$$

$$\dot{a}_2=-\varepsilon\mu_2 a_2+\frac{\varepsilon\alpha_2 a_1^2}{4\omega_2}\sin\varphi_2$$

$$a_2\dot{\varphi}_2=a_2(\omega_2-2\omega_1)+\left(\frac{\varepsilon\alpha_2 a_1^2}{4\omega_2}-\frac{\varepsilon\alpha_1 a_2^2}{2\omega_1}\right)\cos\varphi_2-\frac{\varepsilon\alpha_1 a_2 F_2}{\omega_1\omega_2^2}\cos\Omega t$$

An example of time series associated to variable x_1 is presented in fig. 4. Because the period $T=2\pi/\Omega$ of the stationary motions is much larger than in the previous case, we extended the integration time to 10000 u.t. instead 500 u.t.

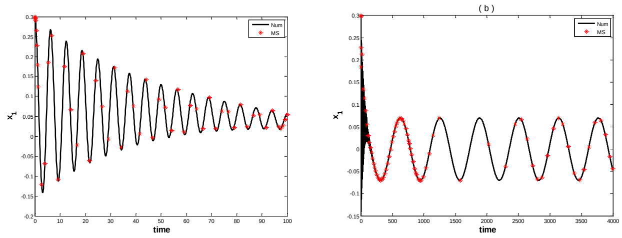

Fig. 4 *Time history for component* x_1 *and parameters values* $\varepsilon=0.1$, $\alpha_1=1$, $\alpha_2=2$, $\mu_1=0.25$, $\mu_2=0.5$, $\omega_1=1$, $\omega_2=2.03$, $\Omega=0.01$ $F_1=0.07$ *and* $F_2=0.05$ *: a) transition period; b) steady – state solution.*

From (3) we conclude that the roll and pitch modes amplitudes do not depend on Ω. Indeed, fig. 5 shows that the long term behavior of system (1) is characterized by amplitudes almost independent on Ω and proportional with F_1 and F_2, respectively.

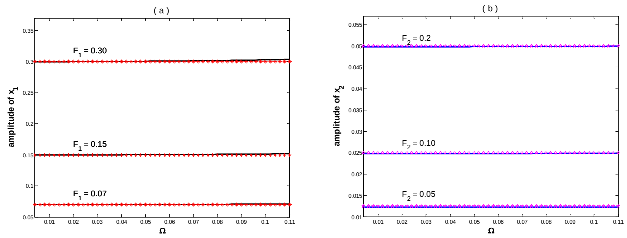

Fig. 5 *Roll and pitch amplitudes as a function of external frequency for constant parameter values* $\varepsilon=0.1$, $\alpha_1=1$, $\alpha_2=2$, $\mu_1=0.25$, $\mu_2=0.5$, $\omega_1=1$, $\omega_2=2.03$ *and different forcing amplitudes .*
a) $F_1\in\{0.07,0.15,0.3\}$ *and* $F_2=0.05$ *;*
b) $F_1=0.07$ *and* $F_2\in\{0.05,0.1,0.2\}$.

2.3. The case $\Omega\approx\omega_1/2$

According to [8], the first-order approximate solution for system (1) is as follows

$$x_1=a_1\cos(2\Omega t-\varphi_1)+\Lambda_1\cos\Omega t$$
$$x_2=a_2\cos(4\Omega t+\varphi_2-2\varphi_1)+\Lambda_2\cos\Omega t \qquad (5)$$

with amplitudes and phases provided by the differential system of equations

$$\dot{a}_1=-\varepsilon\mu_1 a_1-\frac{\varepsilon\alpha_1 a_1 a_2}{4\omega_1}\sin\varphi_2-\frac{\varepsilon\alpha_1\Lambda_1\Lambda_2}{4\omega_1}\sin\varphi_1$$

$$a_1\dot{\varphi}_1=a_1(2\Omega-\omega_1)-\frac{\varepsilon\alpha_1\Lambda_1\Lambda_2}{4\omega_1}\cos\varphi_1-\frac{\varepsilon\alpha_1 a_1 a_2}{4\omega_1}\cos\varphi_2 \qquad (6)$$

$$\dot{a}_2=-\varepsilon\mu_2 a_2+\frac{\varepsilon\alpha_2 a_1^2}{4\omega_2}\sin\varphi_2$$

$$a_2\dot{\varphi}_2=a_2(\omega_2-2\omega_1)-\frac{\varepsilon\alpha_1 a_2\Lambda_1\Lambda_2}{2a_1\omega_1}\cos\varphi_1+\left(\frac{\varepsilon\alpha_2 a_1^2}{4\omega_2}-\frac{\varepsilon\alpha_1 a_2^2}{2\omega_1}\right)\cos\varphi_2$$

Fig. 6 contrasts the time series for rolling, given either by numerical integration of system (1) or by multiple scales method, for a set of parameters selected without order violations. As seen in the figure, the two solutions are again in excellent agreement.

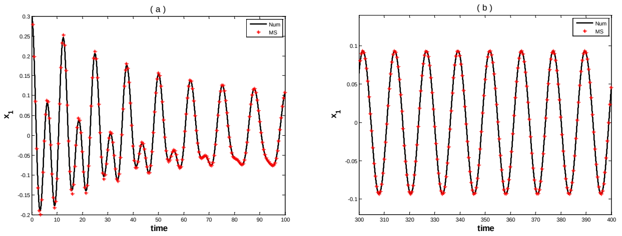

Fig. 6 *Time history for component* x_1 *and parameters values* $\varepsilon = 0.1$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\mu_1 = 0.25$, $\mu_2 = 0.5$, $\omega_1 = 1$, $\omega_2 = 2.03$, $\Omega = 0.5$ $F_1 = 0.07$ *and* $F_2 = 0.05$ *: a) transition period; b) steady – state solution.*

As concerns the steady-state amplitudes for the two modes of oscillation, at low level of forcing none difference is observed when compared with non-resonant case. Some irrelevant jumps are registered around $\Omega = 0.5$ for large forcing, as shown in fig. 7.

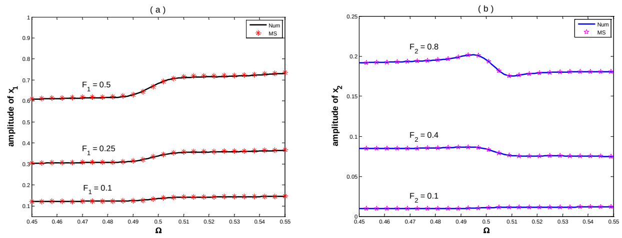

Fig. 7 *Roll and pitch amplitudes as a function of external frequency for constant parameter values* $\varepsilon = 0.1$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\mu_1 = 0.25$, $\mu_2 = 0.5$, $\omega_1 = 1$, $\omega_2 = 2.03$ *and different forcing amplitudes .*
a) $F_1 \in \{0.1, 0.25, 0.5\}$ *and* $F_2 = 0.4$ *;*
b) $F_1 = 0.5$ *and* $F_2 \in \{0.1, 0.4, 0.8\}$.

2.4. The case $\Omega \approx \omega_1$

The system (1) evolves according to the laws

$$x_1 = a_1 \cos(\Omega t - \varphi_1), x_2 = a_2 \cos(2\Omega t + \varphi_2 - 2\varphi_1) + \Lambda_2 \cos \Omega t \quad (7)$$

where the amplitudes $a_n, n = 1,2$, and phases $\varphi_n, n = 1,2$ are the steady-state solutions of the following system

$$\dot{a}_1 = -\varepsilon \mu_1 a_1 - \frac{\varepsilon \alpha_1 a_1 a_2}{4\omega_1} \sin \varphi_2 + \frac{F_1}{2\omega_1} \sin \varphi_1$$

$$a_1 \dot{\varphi}_1 = a_1(\Omega - \omega_1) + \frac{F_1}{2\omega_1} \cos \varphi_1 - \frac{\varepsilon \alpha_1 a_1 a_2}{4\omega_1} \cos \varphi_2 \quad (8)$$

$$\dot{a}_2 = -\varepsilon \mu_2 a_2 + \frac{\varepsilon \alpha_2 a_1^2}{4\omega_1} \sin \varphi_2$$

$$a_2 \dot{\varphi}_2 = a_2(\omega_2 - 2\omega_1) + \frac{a_2 F_1}{a_1 \omega_1} \cos \varphi_1 + \left(\frac{\varepsilon \alpha_2 a_1^2}{4\omega_2} - \frac{\varepsilon \alpha_1 a_2^2}{2\omega_1} \right) \cos \varphi_2$$

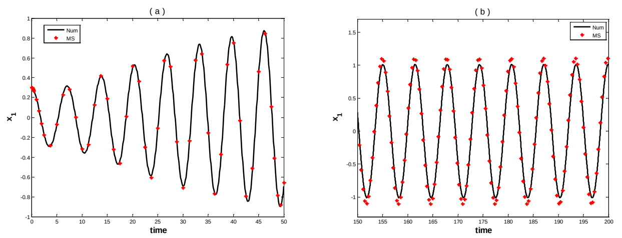

Fig. 8 *Time history for component* x_1 *and parameters values* $\varepsilon = 0.1$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\mu_1 = 0.25$, $\mu_2 = 0.5$, $\omega_1 = 1$, $\omega_2 = 2.03$, $\Omega = 0.98$ $F_1 = 0.07$ *and* $F_2 = 0.05$ *: a) transition period; b) steady – state solution.*

For the primary resonance $\Omega \approx \omega_1$ even small forcing leads to high amplitudes of oscillation. Thus, fig. 8 illustrates this remark by showing the time evolution of variable x_1. The transition period is characterized by a fast growth of rolling and pitching amplitudes and a good agreement between numerical and analytical solutions. On the other hand, a difference of about 5-20% between the two solutions is recorded in the long-term behavior, depending on the external forcing (see also fig.9).

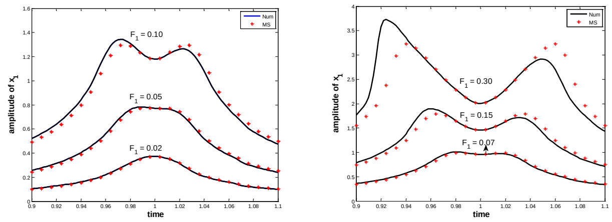

Fig. 9 *Roll amplitudes as a function of external frequency for constant parameter values* $\varepsilon = 0.1$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\mu_1 = 0.25$, $\mu_2 = 0.5$, $\omega_1 = 1$, $\omega_2 = 2.03$ *and different forcing amplitudes .*
Left: $F_1 \in \{0.02, 0.05, 0.1\}$ *and* $F_2 = 0.25$ *;*
Right: $F_1 \in \{0.07, 0.15, 0.3\}$ *and* $F_2 = 0.5$ *;.*

In the analyzed case, the roll mode is less affected than the pitch one by small changes of the ratio ω_2 / ω_1. Surprisingly, the roll amplitudes in the neighborhood of the external resonance condition $\Omega \approx \omega_1$ are smaller for pure resonance $\omega_2 / \omega_1 = 2$ than for $\omega_2 / \omega_1 = 2.1$ or 1.8. However, in so far as the condition $\omega_2 / \omega_1 \approx 2$ is fulfilled the pitch amplitudes get higher. Some frequency – response curves showing these features are presented in fig. 10, where the numerical solution is plotted only.

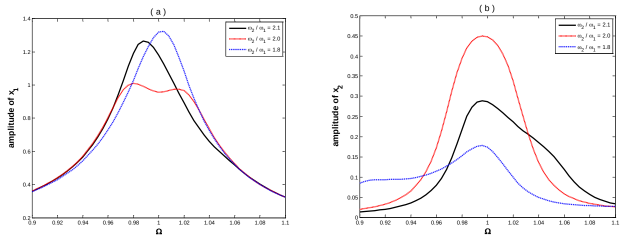

Fig. 10 *Roll and pitch amplitudes as a function of external frequency for constant parameter values* $\varepsilon = 0.1$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\mu_1 = 0.25$, $\mu_2 = 0.5$, $\omega_1 = 1$, $F_1 = 0.07$, $F_2 = 0.05$ *and different ratios* ω_2 / ω_1 :
a) roll mode; b) pitch mode.

The case $\Omega \approx \omega_2$

In the case of this primary resonance, the long-term behavior of system (1) is governed by the laws

$$x_1 = a_1 \cos\left(\frac{\Omega t - \varphi_1 - \varphi_2}{2} \right) + \Lambda_1 \cos \Omega t, x_2 = a_2 \cos(\Omega t - \varphi_1) \quad (9)$$

while the transient towards the steady-state is described by the differential equations

$$\dot{a}_1 = -\varepsilon \mu_1 a_1 - \frac{\varepsilon \alpha_1 a_1 a_2}{4\omega_1} \sin \varphi_2$$

$$a_1 \dot{\varphi}_1 = a_1(\Omega - \omega_2) + \frac{F_2 a_1}{2\omega_2 a_2} \cos \varphi_1 - \frac{\varepsilon \alpha_2 a_1^3}{4\omega_2 a_2} \cos \varphi_2 \quad (10)$$

$$\dot{a}_2 = -\varepsilon \mu_2 a_2 + \frac{F_2}{2\omega_2} \sin \varphi_1 + \frac{\varepsilon \alpha_2 a_1^2}{4\omega_2} \sin \varphi_2$$

$$a_2 \dot{\varphi}_2 = a_2(\omega_2 - 2\omega_1) - \frac{F_2}{2\omega_2} \cos \varphi_1 + \left(\frac{\varepsilon \alpha_2 a_1^2}{4\omega_2} - \frac{\varepsilon \alpha_1 a_2^2}{2\omega_1} \right) \cos \varphi_2$$

As is it pointed out in [8], for small forcing amplitudes F_2 and arbitrary F_1 the roll mode is not excited at all, while the pitch amplitudes are monotonic increasing quantities in F_2. After the forcing F_2 reaches a critical value, the pitch amplitudes remain almost constant and the supplementary energy introduced in the system is directed towards the roll mode and produces a fast growth of amplitude a_1 (see fig. 11a). This transfer of energy is realized only if the condition $\Omega \approx \omega_2$ is satisfied (see fig. 11a).

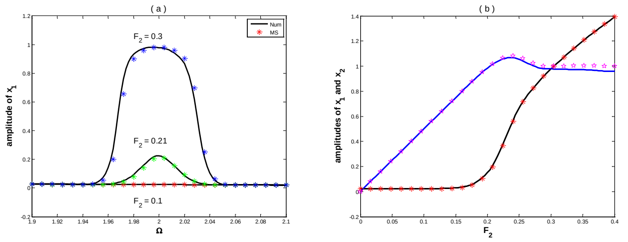

Fig. 11 *a) Roll amplitudes as a function of external frequency for constant parameter values* $\varepsilon = 0.1$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\mu_1 = 0.25$, $\mu_2 = 0.5$, $\omega_1 = 1$, $\omega_2 = 2$, $F_1 = 0.07$ *and different forcing* F_2; *b) roll and pitch amplitudes versus* F_2 *for* $\Omega = 2$.

A typical time series for the x_1 component, both for transient and stationary periods, is reported in fig. 12. This time, some disagreement between the numerical and analytical solutions is observed in the transition stage.

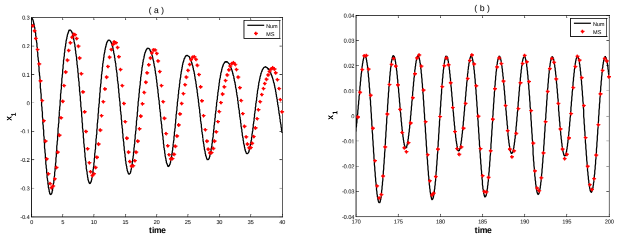

Fig. 12 *Time history for component* x_1 *and parameters values* $\varepsilon = 0.1$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\mu_1 = 0.25$, $\mu_2 = 0.5$, $\omega_1 = 1$, $\omega_2 = 2$, $\Omega = 2$, $F_1 = 0.07$ *and* $F_2 = 0.2$ *: a) transition period; b) steady – state solution.*

The steady-state amplitudes provided by the two methods match extremely or , at least, pretty well even for large forcing, as shown in fig. 13.

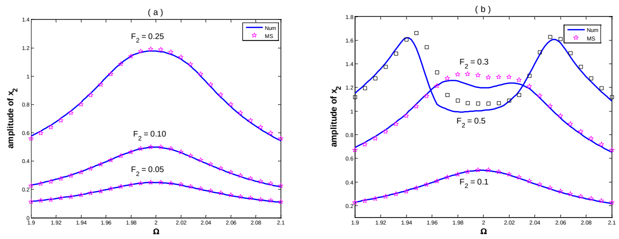

Fig. 13 *Pitch amplitudes as a function of external frequency for constant parameter values* $\varepsilon = 0.1$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\mu_1 = 0.25$, $\mu_2 = 0.5$, $\omega_1 = 1$, $\omega_2 = 2$ *and different forcing amplitudes .*
a) $F_2 \in \{0.05, 0.10, 0.25\}$ *and* $F_1 = 0.07$;
b) $F_2 \in \{0.1, 0.3, 0.5\}$ *and* $F_1 = 0.3$.

The case $\Omega \approx \omega_1 + \omega_2$ is treated in detail in [7], such that its analysis is omitted here. The results are somewhat similar to those in the case $\Omega \approx \omega_1 / 2$.

3. Conclusions

In the paper, the nonlinear responses of a two-degrees-of-freedom model for pitch and roll ship motions have been studied numerically for the particular case in which the pitch frequency is almost twice the roll frequency. Four resonant cases and a non-resonant one have been analyzed and the obtained numerical solution was compared with its analytical counterpart derived in a companion paper. Using time series plots and frequency-amplitude curves, we have proved that the two solutions match well if the system's parameters were selected without order violations, both for transition and steady-state periods. The presented results are in good agreement with the available published studies on the same topic.

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