SIMULTANEOUS RESONANCE CASES IN A PITCH – ROLL SHIP MODEL. PART 1: FIRST – ORDER APPROXIMATE SOLUTIONS

Assoc. Prof. Dr. Deleanu D.
Faculty of Naval Electro mechanics - Maritime University of Constanta, Romania
dumitrudeleanu@yahool.com

Abstract: *In the paper, a two-degrees-of-freedom ship model with quadratic coupled pitch and roll modes under sinusoidal harmonic excitation is considered. The Multiple Scales perturbation technique is applied to yield the first-order expansions for the special internal resonant case where the pitch frequency is twice the roll frequency. Increasing the wave frequency from zero to infinity, five resonant situations are detected. For each case, the governing equations for the transition towards the steady-state solutions, the first-order approximations for these solutions and the frequency-amplitude relationships are presented. A detailed analysis is performed only for the case where the excitation frequency is half of the roll frequency. The reliability of the analytical results derived in the paper is checked in a companion contribution by comparison with numerical solutions.*

Keywords: SHIP ROLLING AND PITCHING, MULTIPLE SCALES, INTERNAL AND EXTERNAL RESONANCES

1. Introduction

A ship in waves, seen as a floating rigid body, has six degrees of freedom (called surge, sway, heave, roll, yaw and pitch), so its motion is extremely complicated and difficult to predict. Accurate calculation of ship – wave hydrodynamic interactions leads to strongly non – linear models, whose analysis is almost very cumbersome and has a computational cost remarkably high.

Between the six oscillatory motions of the ship, the most obvious are rolling and pitching. Roll is of maximum interest because with a typical hull-form, it is the least damped of the motions, hence the roll angle amplitude can be large enough to affect the ship's stability and even to capsize it. Pitch is also important because in heavy sea conditions it can cause the bow to come out of the water, then slam it hard in the water, and introducing enormous efforts in the ship structure. The other motions are typically heavily damped [1].

Various models with two or three degrees of freedom, including rolling and pitching, have been proposed by different authors. Thus, Ghadimi *et al*. have developed a model for the analysis of simultaneous heave, pitch and roll motions of planning vessels in regular waves [2]. Haddara and Xu have investigated the free response of a heaving and pitching ship from its stationary response to random waves [3]. Eissa *et al*. have modeled the interaction of heave and roll by a mass-spring-pendulum system where the effect of waves was included by a periodic forcing term [4]. Pan and co-workers have studied non-stationary responses of a ship model with nonlinearity coupled pitch and roll under modulated excitation. In their model, the differential equations of motion for a ship restrained to pitch and roll modes are as follows

$$\begin{aligned} &\ddot{x}_1+2\varepsilon\mu_1\dot{x}_1+\omega_1^2x_1+\varepsilon\alpha_1x_1x_2=F_1\cos\Omega t\\ &\ddot{x}_2+2\varepsilon\mu_2\dot{x}_2+\omega_2^2x_2+\varepsilon\alpha_2x_1^2=F_2\cos\Omega t \end{aligned} \tag{1}$$

where x_1 and x_2 are the roll and pitch modal amplitudes, μ_1 and μ_2 the modal damping coefficients, ω_1 and ω_2 the natural angular frequencies, Ω the excitation (wave) frequency, F_1 and F_2 the excitation force amplitudes, α_1 and α_2 the coefficients of nonlinear terms, and ε a small parameter [5, 6]. Finally, the dots stand for the differentiation with respect to time *t* and all the coefficients in (1) depend on fluid parameters, ship's characteristics, etc. Kamel has applied the multiple scales method for finding the approximate response of system (1) in the special case of internal resonance $\omega_2\approx 2\omega_1$ associated with the combined external resonance $\omega_1+\omega_2=\Omega$ [7]. Continuing the work of Kamel for giving a picture of the various possible cases, Deleanu has investigated the primary resonance $\omega_1\approx\Omega$ [8].

In the paper we retain the condition that the pitch frequency is twice the roll frequency and make a complete analysis of the different scenarios to appear when the excitation frequency is increased from zero to infinity. We describe in some detail the solution for the secondary resonance $\Omega\approx\omega_1/2$ and present the first – order approximations for the other cases of interest, namely $\Omega\approx 0$, $\Omega\approx\omega_1$, $\Omega\approx\omega_2$, $\Omega\approx\omega_1+\omega_2$ and Ω far from the mentioned values. In a companion contribution, we verify the accuracy of the analytical approximations derived in the paper by contrasting them with the numerical solutions [9].

2. First – order uniform expansions with Multiple Scales method

Because the governing system (1) is weakly nonlinear, we employ the Multiple Scales method to determine its approximate solutions. Thus, we consider two time scales, namely the fast time $T_0=t$ and the slow time $T_1=\varepsilon t$, and expand the dependent variables x_1, x_2 and their derivatives in power series in the small parameter ε

$$\begin{aligned} &x_1=x_{10}(T_0,T_1)+\varepsilon x_{11}(T_0,T_1)+...\\ &x_2=x_{20}(T_0,T_1)+\varepsilon x_{21}(T_0,T_1)+...\\ &\frac{d}{dt}=D_0+\varepsilon D_1+...,\ \frac{d^2}{dt^2}=D_0^2+2\varepsilon D_0D_1+... \end{aligned} \tag{2}$$

where $D_i=\dfrac{\partial}{\partial T_i}$, $D_i^2=\dfrac{\partial^2}{\partial T_i^2}$, and $D_iD_j=\dfrac{\partial^2}{\partial T_i\,\partial T_j}, i,j\in\{1,2\}$.

In what follows, we treat separately the six mentioned situations beginning with the external resonant cases, and closing with the non-resonant one. For the sake of space, an extended solution of the problem is presented only for the first analyzed case.

The case $\Omega\approx\omega_1/2$

This case corresponds to a secondary resonance because a small - divisor term appears in the second order term, x_{11} [8]. Substituting (2) into (1) and equating coefficients of similar powers of ε, one obtains the following set of linear partial differential equations which can be solved successively

$$\varepsilon^0\begin{cases} D_0^2x_{10}+\omega_1^2x_{10}=F_1\cos\Omega T_0\\ D_0^2x_{20}+\omega_2^2x_{20}=F_2\cos\Omega T_0 \end{cases} \tag{3}$$

$$\varepsilon^1\begin{cases} D_0^2x_{11}+\omega_1^2x_{11}=-2D_0D_1x_{10}-2\mu_1D_0x_{10}-\alpha_1x_{10}x_{20}\\ D_0^2x_{21}+\omega_2^2x_{21}=-2D_0D_1x_{20}-2\mu_2D_0x_{20}-\alpha_2x_{10}^2 \end{cases} \tag{4}$$

The first order solutions of (3) have the form

$$x_{10} = A_{10}(T_1)\exp(i\omega_1 T_0) + \frac{\Lambda_1}{2}\exp(i\Omega T_0) + cc$$
$$x_{20} = A_{20}(T_1)\exp(i\omega_2 T_0) + \frac{\Lambda_2}{2}\exp(i\Omega T_0) + cc \qquad (5)$$

where cc stand for the complex conjugates of the preceding terms and $\Lambda_n = F_n / (\omega_n^2 - \Omega^2), n = 1,2$. To express the closeness of Ω to $\omega_1/2$ and of ω_2 to $2\omega_1$ we introduce the detuning parameters σ_1 and σ_2 as follows

$$2\Omega = \omega_1 + \varepsilon\sigma_1, \omega_2 = 2\omega_1 + \varepsilon\sigma_2 \qquad (6)$$

Introducing (5) and (6) into (4), one obtains

$$\left(D_0^2+\omega_1^2\right)x_{11} = \Big[-2i\omega_1(D_1 A_{10} + \mu_1 A_{10}) - \frac{\alpha_1\Lambda_1\Lambda_2}{2}\exp(i\sigma_1 T_1) - $$
$$-\alpha_1 \bar{A}_{10} A_{20}\exp(i\sigma_2 T_1)\Big]\exp(i\omega_1 T_0) + NST_1 \qquad (7)$$
$$\left(D_0^2+\omega_2^2\right)x_{21} = \left[-2i\omega_2(D_1 A_{20} + \mu_2 A_{20}) - \alpha_2 A_{10}^2\exp(-i\sigma_2 T_1)\right]$$
$$\cdot\exp(i\omega_2 T_0) + NST_2$$

where $NST_{1,2}$ stand for the terms which do not produce secular terms. The later, in (7), will vanish if and only if the coefficients of $\exp(i\omega_n T_0), n = 1,2$, are equal to zero

$$-2i\omega_1(D_1 A_{10} + \mu_1 A_{10}) - \alpha_1 \bar{A}_{10} A_{20} e^{i\sigma_2 T_1} - \frac{\alpha_1\Lambda_1\Lambda_2}{2} e^{i\sigma_1 T_1} = 0$$
$$-2i\omega_2(D_1 A_{20} + \mu_2 A_{20}) - \alpha_2 A_{10}^2\exp(-i\sigma_2 T_1) = 0 \qquad (8)$$

We consider now the polar forms of the functions $A_{n0}, n = 1,2$

$$A_{n0}(T_1) = \frac{1}{2} a_n(T_1)\exp(i\eta_n(T_1)), n = 1, 2 \qquad (9)$$

Inserting (9) into (8) and separating real and imaginary parts gives the following first order differential system of equations

$$a_1' = -\mu_1 a_1 - \frac{\alpha_1 a_1 a_2}{4\omega_1}\sin\varphi_2 - \frac{\alpha_1\Lambda_1\Lambda_2}{4\omega_1}\sin\varphi_1$$
$$a_1\eta_1' = \frac{\alpha_1 a_1 a_2}{4\omega_1}\cos\varphi_2 + \frac{\alpha_1\Lambda_1\Lambda_2}{4\omega_1}\cos\varphi_1$$
$$a_2' = -\mu_2 a_2 + \frac{\alpha_2 a_1^2}{4\omega_2}\sin\varphi_2 \qquad (10)$$
$$a_2\eta_2' = \frac{\alpha_2 a_1^2}{4\omega_2}\cos\varphi_2$$

where $\varphi_1 = \sigma_1 T_1 - \eta_1, \varphi_2 = \sigma_2 T_1 + \eta_2 - 2\eta_1$, and primes denote the differentiation with respect to slow time T_1.

Observing that $\varphi_1' = \sigma_1 - \eta_1', \varphi_2' = \sigma_2 + \eta_2' - 2\eta_1'$, the second and the fourth equations (10) may be rewritten as

$$a_1\varphi_1' = a_1\sigma_1 - \frac{\alpha_1\Lambda_1\Lambda_2}{4\omega_1}\cos\varphi_1 - \frac{\alpha_1 a_1 a_2}{4\omega_1}\cos\varphi_2$$
$$a_2\varphi_2' = a_2\sigma_2 - \frac{\alpha_1 a_2\Lambda_1\Lambda_2}{2a_1\omega_1}\cos\varphi_1 + \left(\frac{\alpha_2 a_1^2}{4\omega_2} - \frac{\alpha_1 a_2^2}{2\omega_1}\right)\cos\varphi_2 \qquad (11)$$

Thus, the first – order approximate solution for the system (1) is as follows

$$x_1 = a_1\cos(\omega_1 T_0 + \eta_1) + \Lambda_1\cos\Omega T_0 =$$
$$= a_1\cos(2\Omega t - \varphi_1) + \Lambda_1\cos\Omega t$$
$$x_2 = a_2\cos(\omega_2 T_0 + \eta_2) + \Lambda_2\cos\Omega T_0 = \qquad (12)$$
$$= a_2\cos(4\Omega t + \varphi_2 - 2\varphi_1) + \Lambda_2\cos\Omega t$$

where the amplitudes $a_n, n = 1,2$, and phases $\varphi_n, n = 1,2$, are given by (10) and (11) after returning to the normal time *t*. The functions $a_n, \varphi_n, n = 1,2$ tend to constant values as time *t* is large enough [10-12]. If we insert these values into (12), we obtain the so - called *steady – state solutions*, which are periodic with frequencies 2Ω and 4Ω. To obtain the steady-state solutions we have two choices. First, we can integrate for a large enough period of time. Second, we can use the fact that $a_n, \varphi_n, n = 1,2$, are constants, set $\dot{a}_n = \dot{\varphi}_n = 0, n = 1,2$, in (11) and solve for $\sin\varphi_n$ and $\cos\varphi_n$. But $\sin^2\varphi_n + \cos^2\varphi_n = 1$, so we get a system of equations in the amplitudes $a_n, n = 1,2$, called *frequency – response equations*

$$\mu_2^2 + (2\sigma_1 - \sigma_2)^2 = \left(\frac{\alpha_1 a_1^2}{4\omega_2 a_2}\right)^2 \qquad (13)$$
$$\left(\mu_1 + \mu_2\frac{\alpha_1}{\alpha_2}\frac{\omega_2}{\omega_1}\frac{a_2^2}{a_1^2}\right)^2 + \left(\sigma_1 + (\sigma_2 - 2\sigma_1)\frac{\alpha_1}{\alpha_2}\frac{\omega_2}{\omega_1}\frac{a_2^2}{a_1^2}\right)^2 = \left(\frac{\alpha_1\Lambda_1\Lambda_2}{4\omega_1 a_1}\right)^2$$

Substituting a_2^2/a_1^2 from the first equation (13) and inserting into the second, yields a cubic equation in a_1^2

$$\frac{\alpha_1^2\alpha_2^2}{256\omega_1^2\omega_2^2\left(\mu_2^2 + (\sigma_2 - 2\sigma_1)^2\right)}a_1^6 + \frac{\alpha_1\alpha_2\left(\mu_1\mu_2 + \sigma_1(\sigma_2 - 2\sigma_1)\right)}{8\omega_1\omega_2\left(\mu_2^2 + (\sigma_2 - 2\sigma_1)^2\right)}a_1^4 +$$
$$+\left(\mu_1^2 + \sigma_1^2\right)a_1^2 - \left(\frac{\alpha_1\Lambda_1\Lambda_2}{4\omega_1}\right)^2 = 0 \qquad (14)$$

Generally, it has only one acceptable solution a_1 which increases almost linearly with f_1.

The case $\Omega \approx 0$

The smallness of external frequency may be explicitly introduced in the equations by means of a detuning parameter, σ, so that $\Omega = \varepsilon\sigma$. But now $\cos\Omega t = \cos\sigma T_1$, such that the solution of (3) is

$$x_{10} = A_{10}(T_1)\exp(i\omega_1 T_0) + \frac{F_1}{2\omega_1^2}\exp(i\sigma T_1) + cc$$
$$x_{20} = A_{20}(T_1)\exp(i\omega_2 T_0) + \frac{F_2}{2\omega_2^2}\exp(i\sigma T_1) + cc \qquad (15)$$

Inserting it in (4) and preventing the secular terms, one gets the system describing the motion in the transient stage

$$a_1' = -\mu_1 a_1 - \frac{\alpha_1 a_1 a_2}{4\omega_1}\sin\varphi_2$$
$$a_1\varphi_1' = \frac{\alpha_1 a_1 a_2}{4\omega_1}\cos\varphi_2 + \frac{\alpha_1 a_1 F_2}{2\omega_1\omega_2^2}\cos\sigma T_1$$
$$a_2' = -\mu_2 a_2 + \frac{\alpha_2 a_1^2}{4\omega_2}\sin\varphi_2 \qquad (16)$$
$$a_2\varphi_2' = a_2\sigma_2 + \left(\frac{\alpha_2 a_1^2}{4\omega_2} - \frac{\alpha_1 a_2^2}{2\omega_1}\right)\cos\varphi_2 - \frac{\alpha_1 a_2 F_2}{\omega_1\omega_2^2}\cos\sigma T_1$$

Comparing the first and the third equation, we easily observe that the only steady-state solution is $a_1 = a_2 = 0$. Thus, the long-term behavior is described by the laws

$$x_1 = \frac{F_1}{\omega_1^2}\cos\Omega t, x_2 = \frac{F_2}{\omega_2^2}\cos\Omega t \qquad (17)$$

The case $\Omega \approx \omega_1$

This time we face with a primary resonance because the small-divisor terms appear firstly in the term x_{10}. The case was discussed extensively in [8] so we reproduce here only the main results. Thus, the first-order approximation for the solution of (1) is written as

$$x_1 = a_1\cos(\Omega t - \varphi_1),\ x_2 = a_2\cos(2\Omega t + \varphi_2 - 2\varphi_1) + \Lambda_2\cos\Omega t \quad (18)$$

where the amplitudes $a_n, n = 1,2$, and phases $\varphi_n, n = 1,2$ are the steady-state solutions of the following system

$$\begin{aligned}
\dot a_1 &= -\varepsilon\mu_1 a_1 - \frac{\varepsilon\alpha_1 a_1 a_2}{4\omega_1}\sin\varphi_2 + \frac{F_1}{2\omega_1}\sin\varphi_1 \\
a_1\dot\varphi_1 &= a_1(\Omega - \omega_1) + \frac{F_1}{2\omega_1}\cos\varphi_1 - \frac{\varepsilon\alpha_1 a_1 a_2}{4\omega_1}\cos\varphi_2 \\
\dot a_2 &= -\varepsilon\mu_2 a_2 + \frac{\varepsilon\alpha_2 a_1^2}{4\omega_1}\sin\varphi_2 \\
a_2\dot\varphi_2 &= a_2(\omega_2 - 2\omega_1) + \frac{a_2 F_1}{a_1\omega_1}\cos\varphi_1 + \left(\frac{\varepsilon\alpha_2 a_1^2}{4\omega_2} - \frac{\varepsilon\alpha_1 a_2^2}{2\omega_1}\right)\cos\varphi_2
\end{aligned} \quad (19)$$

It is almost identical to (10), the only difference being the replacement of F_1 with $-\frac{\varepsilon\alpha_1\Lambda_1\Lambda_2}{2}$. The same change is necessary in frequency-response equations' writing.

The case $\Omega \approx \omega_2$

In the case of this primary resonance, the system (1) evolves according to the laws

$$x_1 = a_1\cos\left(\frac{\Omega t - \varphi_1 - \varphi_2}{2}\right) + \Lambda_1\cos\Omega t,\ x_2 = a_2\cos(\Omega t - \varphi_1) \quad (20)$$

The transient towards the steady-state is governed by the differential equations

$$\begin{aligned}
\dot a_1 &= -\varepsilon\mu_1 a_1 - \frac{\varepsilon\alpha_1 a_1 a_2}{4\omega_1}\sin\varphi_2 \\
a_1\dot\varphi_1 &= a_1(\Omega - \omega_2) + \frac{F_2 a_1}{2\omega_2 a_2}\cos\varphi_1 - \frac{\varepsilon\alpha_2 a_1^3}{4\omega_2 a_2}\cos\varphi_2 \\
\dot a_2 &= -\varepsilon\mu_2 a_2 + \frac{F_2}{2\omega_2}\sin\varphi_1 + \frac{\varepsilon\alpha_2 a_1^2}{4\omega_2}\sin\varphi_2 \\
a_2\dot\varphi_2 &= a_2(\omega_2 - 2\omega_1) - \frac{F_2}{2\omega_2}\cos\varphi_1 + \left(\frac{\varepsilon\alpha_2 a_1^2}{4\omega_2} - \frac{\varepsilon\alpha_1 a_2^2}{2\omega_1}\right)\cos\varphi_2
\end{aligned} \quad (21)$$

A careful inspection of (21) reveals an interesting behavior, suggested by Nayfeh and co-workers [11]. If the excitation amplitude F_2 is smaller than the critical value

$$F_{2,cr} = \frac{8\omega_1\omega_2}{\alpha_1}\sqrt{\mu_1^2 + \left(\frac{\Omega - 2\omega_1}{2\varepsilon}\right)^2}\sqrt{\varepsilon^2\mu_2^2 + (\omega_2 - \Omega)^2} \quad (22)$$

the roll amplitude a_1 is null, while a_2 increases directly proportional with F_2,

$$a_2 = \frac{F_2}{2\omega_2\sqrt{\varepsilon^2\mu_2^2 + (\omega_2 - \Omega)^2}} \quad (23)$$

As F_2 increases over $F_{2,cr}$, the pitch mode is saturated (amplitude a_2 remains constant), and the extra energy introduced in system contributes to the roll mode. For this second stage, the frequency – response equations have the form

$$\mu_1^2 + \left(\frac{\sigma_1 + \sigma_2}{2}\right)^2 = \left(\frac{\alpha_1 a_2}{4\omega_1}\right)^2 \quad (24)$$

$$\left(\mu_1 + \mu_2\frac{\alpha_1}{\alpha_2}\frac{\omega_2}{\omega_1}\frac{a_2^2}{a_1^2}\right)^2 + \left(\frac{\sigma_1 + \sigma_2}{2} - \sigma_1\frac{\alpha_1}{\alpha_2}\frac{\omega_2}{\omega_1}\frac{a_2^2}{a_1^2}\right)^2 = \left(\frac{\alpha_1 a_2 f_2}{2\alpha_2\omega_1 a_1^2}\right)^2$$

The case $\Omega \approx \omega_1 + \omega_2$

For this compound secondary resonance, Kamel [7] found the first-order solution

$$\begin{aligned}
x_1 &= a_1\cos\left(\frac{\Omega t - \varphi_1 - \varphi_2}{3}\right) + \Lambda_1\cos\Omega t \\
x_2 &= a_2\cos\left(\frac{2\Omega t + \varphi_1 - \varphi_2}{3}\right) + \Lambda_2\cos\Omega t
\end{aligned} \quad (25)$$

with amplitudes and phases given by

$$\begin{aligned}
\dot a_1 &= -\varepsilon\mu_1 a_1 - \frac{\varepsilon\alpha_1 a_1 a_2}{4\omega_1}\sin\varphi_1 - \frac{\varepsilon\alpha_1 a_2\Lambda_1}{4\omega_1}\sin\varphi_2 \\
a_1\dot\varphi_1 &= a_1(\omega_2 - 2\omega_1) + \left(\frac{\varepsilon\alpha_2 a_1^3}{4\omega_2 a_2} - \frac{\varepsilon\alpha_1 a_1 a_2}{2\omega_1}\right)\cos\varphi_1 - \\
&\quad -\left(\frac{\varepsilon\alpha_2 a_1^2\Lambda_1}{2\omega_1 a_2} + \frac{\varepsilon\alpha_1 a_2\Lambda_1}{2\omega_1}\right)\cos\varphi_2 \\
\dot a_2 &= -\varepsilon\mu_2 a_2 + \frac{\varepsilon\alpha_2 a_1^2}{4\omega_2}\sin\varphi_1 - \frac{\varepsilon\alpha_2 a_1\Lambda_1}{2\omega_2}\sin\varphi_2 \\
a_2\dot\varphi_2 &= a_2(\Omega - \omega_1 - \omega_2) - \left(\frac{\varepsilon\alpha_2 a_1^2}{4\omega_2} + \frac{\varepsilon\alpha_1 a_2^2}{4\omega_1}\right)\cos\varphi_1 - \\
&\quad -\left(\frac{\varepsilon\alpha_2 a_1\Lambda_1}{2\omega_2} + \frac{\varepsilon\alpha_1 a_2^2\Lambda_1}{4\omega_1 a_1}\right)\cos\varphi_2\,.
\end{aligned} \quad (26)$$

The frequency-response equations are written as

$$\begin{aligned}
9\mu_1^2 + (\sigma_1 + \sigma_2)^2 &= (a_1 + \Lambda_1)^2\left(\frac{3\alpha_1 a_2}{4\omega_1 a_1}\right)^2 \\
9\mu_2^2 + (2\sigma_1 - \sigma_2)^2 &= (a_1 + 2\Lambda_2)^2\left(\frac{3\alpha_2 a_1}{4\omega_2 a_2}\right)^2
\end{aligned} \quad (27)$$

where $2\Omega = \omega_1 + \omega_2 + \varepsilon\sigma_1, \omega_2 = 2\omega_1 + \varepsilon\sigma_2$.

The case Ω far from $0, \omega_1/2,\ \omega_1, \omega_2$ and $\omega_1 + \omega_2$

The first – order approximations for the solution of (1) are written as

$$\begin{aligned}
x_1 &= a_1\cos(\omega_1 t + \varphi_1) + \Lambda_1\cos\Omega t \\
x_2 &= a_2\cos(2\omega_1 t + \varphi_2 + 2\varphi_1) + \Lambda_2\cos\Omega t
\end{aligned} \quad (28)$$

where the amplitudes $a_n, n = 1,2$, and phases $\varphi_n, n = 1,2$, are obtained from

$$\begin{aligned}
\dot a_1 &= -\varepsilon\mu_1 a_1 - \frac{\varepsilon\alpha_1 a_1 a_2}{4\omega_1}\sin\varphi_2,\ a_1\dot\varphi_1 = \frac{\varepsilon\alpha_1 a_1 a_2}{4\omega_1}\cos\varphi_2 \\
\dot a_2 &= -\varepsilon\mu_2 a_2 + \frac{\varepsilon\alpha_2 a_1^2}{4\omega_2}\sin\varphi_2 \\
a_2\dot\varphi_2 &= a_2(\omega_2 - 2\omega_1) + \left(\frac{\varepsilon\alpha_2 a_1^2}{4\omega_2} - \frac{\varepsilon\alpha_1 a_2^2}{2\omega_1}\right)\cos\varphi_2
\end{aligned} \quad (29)$$

The only steady-state solution is characterized by $a_1 = a_2 = 0$, so the system (1) oscillates in accordance with the laws

$$x_1 = \Lambda_1\cos\Omega t,\ x_2 = \Lambda_2\cos\Omega t \quad (30)$$

Remark: We should observe that the solution (30) is the same as that achieved in the case $\Omega \approx 0$.

3. Conclusions

In the present paper, a mathematical model for the analysis of simultaneous pitch and roll motions of a ship is considered. The coupling between the two modes of oscillation is realized by quadratic terms and the external excitation is of harmonically type. The ratio between pitch and roll frequencies is taken to be two to one. The governing system of differential equations is weakly nonlinear, thus the multiple scales method is applied to yield first-order approximations both for the transient behavior and for the steady state solutions. Five resonant cases and a non-resonant one are presented, part of them already discussed in the available published work and the other part analyzed in some detail by the author. The accuracy of the analytical solutions derived in the paper is checked in a companion paper and a good agreement with the numerical solution is observed.

References

[1] Fan, S., Xia, J., *Simulation of ship motions – coupled heave, pitch and roll,* Centre for Marine Science and Technology, Technical Report no. 35, 2002.

[2] Ghadimi, P., Dashtimanesh, A., Djeddi, S.R., Maghrebi, Y.F., *Development of a mathematical model for simultaneous heave, pitch and roll motions of planning vessel in regular waves,* International Journal of Scientific World, 1(2), p. 44 – 56, 2013.

[3] Haddara, M.R., Xu, J., *On the identification of the ship coupled heave – pitch motions using neural networks,* Ocean Engineering, 26, p. 381 – 400, 1999.

[4] Eissa, M., El-Sera, S., El-Sheikh, M., Sayeda, M., *Stability and primary simultaneous resonance of harmonically excited non-linear spring pendulum system*, Appl. Math. And Computation, 145, p. 421 – 444, 2003.

[5] Pan, R., Davies, H.G., *Non-stationary response of a non-linear coupled pitch-roll ship model under modulated excitation*, J. of Sound and Vibration, 192, p. 669 – 699, 1996.

[6] Pan, R., Davies, H.G., *Response of a nonlinearly coupled pitch – roll ship model under harmonic excitation*, Journal of Nonlinear Dynamics, 2, p. 211-219, 1994.

[7] Kamel, M.M., *Bifurcation analysis of a nonlinear coupled pitch – roll ship*, Mathematics and Computers in Simulation, 73, p. 300 – 308, 2007.

[8] Deleanu, D., *Analysis of a combined case of internal and external resonances for a quadratic coupled pitch – roll ship*, The XXIV International Scientific Technical Conference TRANS’16 MOTAUTO, Burgas, Bulgaria, June 2016.

[9] Deleanu, D., *Simultaneous resonance cases in a pitch-roll ship model. Part 2: Numerical analysis*, Fourth International Scientific Conference “Engineering, Technology, Education, Security”, Veliko Tarnovo, Bulgaria, 01-03.06 2016.

[10] Dumitrache, C.L., Calimanescu, I., Comandar, C., *Naval standard and safety valve design using CAD solutions,* Int. Conference “Advanced Concepts in Mechanical Engineering” ACME 2014, Iasi, Romania, 12-15.10. 2014.

[11] Nayfeh, A.H., Mook, D.T., Marshall, L.R., *Nonlinear coupling of pitch and roll modes in ship motions*, Journal of Hydronautics, 7(4), p. 145-152, 1973.

[12] Nayfeh, A.H., *Introduction to perturbation techniques,* John Wiley and Sons, New York, 1992.