ANALYSIS OF A COMBINED CASE OF INTERNAL AND EXTERNAL RESONANCES FOR A QUADRATIC COUPLED PITCH – ROLL SHIP

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Abstract: *In the paper, a two-degrees-of-freedom ship model with quadratic coupled pitch and roll modes under sinusoidal harmonic excitation is considered. A straightforward expansion allows for obtaining both the resonant values of external excitation frequency and one internal resonance. The Multiple Scales method yields the first-order expansions for the special resonant case where the excitation frequency is close to the roll frequency. The time series and the frequency – amplitude curves provided by numerical integration are contrasted with those given by the perturbation technique for different combination of system's parameters. If the parameters are selected within the pre-ordered range, the results of both methods are in excellent or, at least, in pretty good agreement.*

Keywords: SHIP ROLLING AND PITCHING, MULTIPLE SCALES, INTERNAL AND EXTERNAL RESONANCES

1. Introduction

Ship motion in waves is a strongly non-linear and multivariable dynamic process whose complete description requires mathematical models with six degrees of freedom. Among them, rolling has the biggest influence on ship's stability. For this reason, one degree of freedom models considering the uncoupled roll motion have been widely proposed by the scientific community [1-4] . The models are useful, for example, to obtain insight into the parametric roll resonance phenomenon, but they have too little complexity for describing the entire behavior of the ship. Since the coupling between the roll and other modes of motion was found to be extremely important in the correlation of the theoretical and experimental results, mathematical models following an increasing sophistication and capable of predicting well enough the ship dynamics for some given operational conditions have been suggested. Thus, the analysis of ship dynamics and stability by means of two-degrees-of-freedom models has received considerable attention. For example, Haddara and Xu have investigated the free response of a heaving and pitching ship from its stationary response to random waves [5]. Eissa et al. have modeled the interaction of heave and roll by a mass-spring-pendulum system where the effect of waves was included by a periodic forcing term [6]. Pan and co-workers have studied non-stationary responses of a ship model with nonlinearity coupled pitch and roll under modulated excitation [7]. Using the previously mentioned model, Kamel have applied the Multiple Scales method and obtained the bifurcation response equation near the combination resonance case in the presence of internal resonance [8].

Continuing the work of Kamel for giving a picture of the various possible cases, in the paper we retain the condition that the pitch frequency is twice the roll frequency but we seek for the ship response when the excitation (encounter) frequency is close to the roll frequency. The nonlinear coupled differential equations of motion for a ship restrained to pitch and roll modes are as follows

$$\begin{aligned} &\ddot{x}_1 + 2\varepsilon\mu_1\dot{x}_1 + \omega_1^2 x_1 + \varepsilon\alpha_1 x_1 x_2 = F_1\cos\Omega t \\ &\ddot{x}_2 + 2\varepsilon\mu_2\dot{x}_2 + \omega_2^2 x_2 + \varepsilon\alpha_2 x_1^2 = F_2\cos\Omega t \end{aligned} \tag{1}$$

where x_1 and x_2 are the roll and pitch modal amplitudes, μ_1 and μ_2 the modal damping coefficients, ω_1 and ω_2 the natural angular frequencies, Ω the excitation (wave) frequency, F_1 and F_2 the excitation force amplitudes, α_1 and α_2 the coefficients of nonlinear terms, and ε a small parameter [7]. Finally, the dots stand for the differentiation with respect to time *t*.

2. A Straightforward Expansion for the Coupled Pitch – Roll Equations

In this section we seek for an approximate solution to (1) in the form

$$x_1(t) = x_{10}(t) + \varepsilon x_{11}(t) + ...,\; x_2(t) = x_{20}(t) + \varepsilon x_{21}(t) + ... \tag{2}$$

Substituting (2) into (1) and equating the each of the coefficients of $\varepsilon^n, n = 0,1$, in both parts yields

$$\varepsilon^0:\; \ddot{x}_{10} + \omega_1^2 x_{10} = F_1\cos\Omega t,\quad \ddot{x}_{20} + \omega_2^2 x_{20} = F_2\cos\Omega t \tag{3}$$

$$\begin{aligned} \varepsilon^1:\; &\ddot{x}_{11} + 2\mu_1\dot{x}_{10} + \omega_1^2 x_{11} + \alpha_1 x_{10}x_{20} = 0, \\ &\ddot{x}_{21} + 2\mu_2\dot{x}_{20} + \omega_2^2 x_{21} + \alpha_2 x_{10}^2 = 0 \end{aligned} \tag{4}$$

The general solution of (3) can be obtained as the sum of a homogeneous solution and a particular solution. It can be expressed as

$$\begin{aligned} &x_{10} = a_1\cos(\omega_1 t + \varphi_1) + \Lambda_1\cos\Omega t \\ &x_{20} = a_2\cos(\omega_2 t + \varphi_2) + \Lambda_2\cos\Omega t \end{aligned} \tag{5}$$

where $\Lambda_1 = \dfrac{F_1}{\omega_1^2 - \Omega^2}$ and $\Lambda_2 = \dfrac{F_2}{\omega_2^2 - \Omega^2}$.

Then, (4) becomes

$$\begin{aligned} &\ddot{x}_{11} + \omega_1^2 x_{11} = 2\mu_1\omega_1 a_1\sin(\omega_1 t + \varphi_1) + 2\mu_1\Lambda_1\Omega\sin\Omega t - \\ &-\alpha_1 a_1 a_2\cos(\omega_1 t + \varphi_1)\cos(\omega_2 t + \varphi_2) - \alpha_1 a_1\Lambda_2\cos(\omega_1 t + \varphi_1)\cdot \\ &\cdot\cos\Omega t - \alpha_1 a_2\Lambda_1\cos(\omega_2 t + \varphi_2)\cos\Omega t - \alpha_1\Lambda_1\Lambda_2\cos^2\Omega t \\ &\ddot{x}_{21} + \omega_2^2 x_{21} = 2\mu_2\omega_2 a_2\sin(\omega_2 t + \varphi_2) + 2\mu_2\Lambda_2\Omega\sin\Omega t - \\ &-\frac{\alpha_2}{2}\left(a_1^2 + \Lambda_1^2\right) - \frac{\alpha_2 a_1^2}{2}\cos(2\omega_1 t + 2\varphi_1) - \frac{\alpha_2\Lambda_1^2}{2}\cos 2\Omega t - \\ &-\alpha_2 a_1\Lambda_1\cos((\omega_1 + \Omega)t + \varphi_1) - \alpha_2 a_1\Lambda_1\cos((\omega_1 - \Omega)t + \varphi_1) \end{aligned} \tag{6}$$

Using trigonometric identities and neglecting the homogeneous solution [9, 10], from (6) it follows that

$$\begin{aligned} &x_{11} = -\mu_1 a_1 t\cos(\omega_1 t + \varphi_1) + \frac{\alpha_1 a_1 a_2}{2\omega_2(\omega_2 + 2\omega_1)}\cos((\omega_1 + \omega_2)t + \varphi_1 + \varphi_2) \\ &+\frac{2\mu_1\Lambda_1\Omega}{\omega_1^2 - \Omega^2}\sin\Omega t + \frac{\alpha_1 a_1 a_2}{2\omega_2(\omega_2 - 2\omega_1)}\cos((\omega_1 - \omega_2)t + \varphi_1 - \varphi_2) + \\ &+\frac{\alpha_1 a_1\Lambda_2}{2\Omega(\Omega + 2\omega_1)}\cos((\omega_1 + \Omega)t + \varphi_1) + \frac{\alpha_1\Lambda_1\Lambda_2}{2(\omega_1 + 2\Omega)(2\Omega - \omega_1)}\cos 2\Omega t + \\ &+\frac{\alpha_1 a_1\Lambda_2}{2\Omega(\Omega - 2\omega_1)}\cos((\omega_1 - \Omega)t + \varphi_1) + \frac{\alpha_1 a_2\Lambda_1}{2(\omega_1 + \omega_2 + \Omega)(\omega_2 - \omega_1 + \Omega)}\cdot \\ &\cdot\cos((\omega_2 + \Omega)t + \varphi_2) + \frac{\alpha_1 a_2\Lambda_1}{2(\omega_1 - \omega_2 + \Omega)(\Omega - \omega_1 - \omega_2)}\cdot \end{aligned}$$

$$\cdot\cos((\omega_2-\Omega)t+\varphi_2)-\frac{\alpha_1 a_1 a_2}{2\omega_1^2} \tag{7}$$

$$x_{21}=-\mu_2 a_2 t\cos(\omega_2 t+\varphi_2)+\frac{2\mu_2\Lambda_2\Omega}{\omega_2^2-\Omega^2}\sin\Omega t-\frac{\alpha_2\left(a_1^2+\Lambda_1^2\right)}{2\omega_2^2}+$$

$$+\frac{\alpha_2\Lambda_1^2}{2\left(4\Omega^2-\omega_2^2\right)}\cos 2\Omega t+\frac{\alpha_2 a_1\Lambda_1}{\left((\omega_1+\Omega)^2-\omega_2^2\right)}\cos((\omega_1+\Omega)t+\varphi_1)+$$

$$+\frac{\alpha_2 a_1^2}{2\left(4\omega_1^2-\omega_2^2\right)}\cos(2\omega_1 t+2\varphi_1)+\frac{\alpha_2 a_1\Lambda_1}{\left((\Omega-\omega_1)^2-\omega_2^2\right)}\cos((\omega_1-\Omega)t+\varphi_1)$$

The approximate solution contains both secular $-\mu_n a_n t\sin(\omega_n t+\varphi_n),\ n=1,2$ and small-divisor terms. The latter occur when $\Omega\approx 0,\ \Omega\approx\omega_n,\ \Omega\approx\frac{\omega_n}{2},\ n=1,2$, or $\Omega\approx\omega_1-\omega_2,\ \Omega\approx\omega_2-\omega_1,\ \Omega\approx\omega_1+\omega_2,\ \Omega\approx 2\omega_1$ and $\omega_2\approx 2\omega_1$. These special frequencies are called *resonant frequencies* and in their neighborhood the relationships (7) fail to give an accurate approximation for the solutions of (1). In the next section we apply the Multiple Scales method to determine uniform expansions for system (1) that do not contain secular or small-divisor terms in the special case of internal resonance $\omega_2\approx 2\omega_1$ coupled with the primary resonance $\Omega\approx\omega_1$.

3. First – order uniform expansions for coupled pitch – roll

In the combined case of primary resonance, $\Omega\approx\omega_1$, and internal resonance, $\omega_2\approx 2\omega_1$, the small – divisor terms appear both in the first term, x_{10}, of the straightforward expansion (2) and in the terms x_{11} and x_{21}. Through Λ_1, x_{10} becomes very large as $\Omega\to\omega_1$ and the nonlinear and damping terms cease to be of $O(\varepsilon)$. As pointed out by Nayfeh [9], the ordering of the terms in (1) is rendered invalid. Assuming that the excitation amplitude F_1 is relatively small, we let $F_1=\varepsilon f_1$ and seek for uniformly valid expansions of x_1 and x_2, this time by using Multiple Scales method.

We consider two time scales, namely the fast time $T_0=t$ and the slow time $T_1=\varepsilon t$, and expand the dependent variables x_1, x_2 and their derivatives in power series in the small parameter ε

$$\begin{aligned}&x_1=x_{10}(T_0,T_1)+\varepsilon x_{11}(T_0,T_1)+...\\&x_2=x_{20}(T_0,T_1)+\varepsilon x_{21}(T_0,T_1)+...\\&\frac{d}{dt}=D_0+\varepsilon D_1+...,\ \frac{d^2}{dt^2}=D_0^2+2\varepsilon D_0D_1+...\end{aligned} \tag{8}$$

where $D_i=\frac{\partial}{\partial T_i},\ D_i^2=\frac{\partial^2}{\partial T_i^2}$, and $D_iD_j=\frac{\partial^2}{\partial T_i\,\partial T_j},i,j\in\{1,2\}$.

Substituting (8) into (1) and equating coefficients of like powers of ε yields the following set of linear partial differential equations which can be solved successively

$$\varepsilon^0\begin{cases}D_0^2x_{10}+\omega_1^2x_{10}=0\\D_0^2x_{20}+\omega_2^2x_{20}=F_2\cos\Omega T_0\end{cases} \tag{9}$$

$$\varepsilon^1\begin{cases}D_0^2x_{11}+\omega_1^2x_{11}=-2D_0D_1x_{10}-2\mu_1D_0x_{10}-\alpha_1x_{10}x_{20}+\\\qquad+f_1\cos\Omega T_0\\D_0^2x_{21}+\omega_2^2x_{21}=-2D_0D_1x_{20}-2\mu_2D_0x_{20}-\alpha_2x_{10}^2\end{cases} \tag{10}$$

The first order solutions of (9) have the form

$$\begin{aligned}&x_{10}=A_{10}(T_1)\exp(i\omega_1T_0)+cc\\&x_{20}=A_{20}(T_1)\exp(i\omega_2T_0)+\frac{\Lambda_2}{2}\exp(i\Omega T_0)+cc\end{aligned} \tag{11}$$

where *cc* stand for the complex conjugates of the preceding terms. To express the nearness of Ω to ω_1 and of ω_2 to $2\omega_1$ we introduce the detuning parameters σ_1 and σ_2 as follows

$$\Omega=\omega_1+\varepsilon\sigma_1,\omega_2=2\omega_1+\varepsilon\sigma_2 \tag{12}$$

Inserting (11) and (12) into (10), one obtains

$$\begin{aligned}&\left(D_0^2+\omega_1^2\right)x_{11}=\Big[-2i\omega_1(D_1A_{10}+\mu_1A_{10})+\frac{1}{2}f_1\exp(i\sigma_1T_1)-\\&\qquad-\alpha_1\bar{A}_{10}A_{20}\exp(i\sigma_2T_1)\Big]\exp(i\omega_1T_0)+NST_1\\&\left(D_0^2+\omega_2^2\right)x_{21}=\left[-2i\omega_2(D_1A_{20}+\mu_2A_{20})-\alpha_2A_{10}^2\exp(-i\sigma_2T_1)\right]\\&\qquad\cdot\exp(i\omega_2T_0)+NST_2\end{aligned} \tag{13}$$

where $NST_{1,2}$ stand for the terms which do not produce secular terms. The later, in (13), will vanish if and only if the coefficients of $\exp(i\omega_nT_0),\ n=1,2$, are equal to zero

$$-2i\omega_1(D_1A_{10}+\mu_1A_{10})-\alpha_1\bar{A}_{10}A_{20}\exp(i\sigma_2T_1)+\frac{f_1}{2}\exp(i\sigma_1T_1)=0$$

$$-2i\omega_2(D_1A_{20}+\mu_2A_{20})-\alpha_2A_{10}^2\exp(-i\sigma_2T_1)=0 \tag{14}$$

We consider now the polar forms of the functions $A_{n0},n=1,2$

$$A_{n0}(T_1)=\frac{1}{2}a_n(T_1)\exp(i\eta_n(T_1)),\ n=1,2 \tag{15}$$

Inserting (15) into (14) and separating real and imaginary parts gives the following first order differential system of equations

$$\begin{aligned}&a_1'=-\mu_1a_1-\frac{\alpha_1a_1a_2}{4\omega_1}\sin\varphi_2+\frac{f_1}{2\omega_1}\sin\varphi_1\\&a_1\eta_1'=\frac{\alpha_1a_1a_2}{4\omega_1}\cos\varphi_2-\frac{f_1}{2\omega_1}\cos\varphi_1\\&a_2'=-\mu_2a_2+\frac{\alpha_2a_1^2}{4\omega_2}\sin\varphi_2\\&a_2\eta_2'=\frac{\alpha_2a_1^2}{4\omega_2}\cos\varphi_2\end{aligned} \tag{16}$$

where $\varphi_1=\sigma_1T_1-\eta_1,\varphi_2=\sigma_2T_1+\eta_2-2\eta_1$, and primes denote the differentiation with respect to slow time T_1.

Observing that $\varphi_1'=\sigma_1-\eta_1',\varphi_2'=\sigma_2+\eta_2'-2\eta_1'$, the second and the fourth equations (16) may be rewritten as

$$\begin{aligned}&a_1\varphi_1'=a_1\sigma_1+\frac{f_1}{2\omega_1}\cos\varphi_1-\frac{\alpha_1a_1a_2}{4\omega_1}\cos\varphi_2\\&a_2\varphi_2'=a_2\sigma_2+\frac{a_2f_1}{a_1\omega_1}\cos\varphi_1+\left(\frac{\alpha_2a_1^2}{4\omega_2}-\frac{\alpha_1a_2^2}{2\omega_1}\right)\cos\varphi_2\end{aligned} \tag{17}$$

Thus, the first – order approximate solution for the system (1) is as follows

$$\begin{aligned}&x_1=a_1\cos(\omega_1T_0+\eta_1)=a_1\cos(\Omega t-\varphi_1)\\&x_2=a_2\cos(\omega_2T_0+\eta_2)+\Lambda_2\cos\Omega T_0=\\&\quad=a_2\cos(2\Omega t+\varphi_2-2\varphi_1)+\Lambda_2\cos\Omega t\end{aligned} \tag{18}$$

where the amplitudes $a_n,n=1,2$, and phases $\varphi_n,n=1,2$, are given by (16) and (17) after returning to the normal time *t*.

As we will see in the next section, the functions $a_n,\varphi_n,n=1,2$ tend to constant values as time *t* is large enough. If we insert these values into (18), we obtain the so - called *steady – state solutions*, which are periodic with frequencies Ω and 2Ω.

To obtain the steady-state solutions we have two choices. First, we can integrate for a large enough period of time. Second, we can use the fact that $a_n, \varphi_n, n=1,2$, are constants, set $\dot{a}_n=\dot{\varphi}_n=0, n=1,2$, in (17) and solve for $\sin\varphi_n$ and $\cos\varphi_n$. But $\sin^2\varphi_n+\cos^2\varphi_n=1$, so we get a system of equations in the amplitudes $a_n, n=1,2$, called *frequency – response equations*

$$\mu_2^2+\left(2\sigma_1-\sigma_2\right)^2=\frac{\alpha_2^2 a_1^4}{16\omega_2^2 a_2^2} \qquad (19)$$

$$\left(\mu_1+\mu_2\frac{\alpha_1}{\alpha_2}\frac{\omega_2}{\omega_1}\frac{a_2^2}{a_1^2}\right)^2+\left(\sigma_1+\left(\sigma_2-2\sigma_1\right)\frac{\alpha_1}{\alpha_2}\frac{\omega_2}{\omega_1}\frac{a_2^2}{a_1^2}\right)^2=\left(\frac{f_1}{2\omega_1 a_1}\right)^2$$

Substituting a_2^2/a_1^2 from the first equation (19) and inserting into the second, yields a cubic equation in a_1^2

$$\frac{\alpha_1^2\alpha_2^2}{256\omega_1^2\omega_2^2\left(\mu_2^2+\left(\sigma_2-2\sigma_1\right)^2\right)}a_1^6+\frac{\alpha_1\alpha_2\left(\mu_1\mu_2+\sigma_1\left(\sigma_2-2\sigma_1\right)\right)}{8\omega_1\omega_2\left(\mu_2^2+\left(\sigma_2-2\sigma_1\right)^2\right)}a_1^4+$$

$$+\left(\mu_1^2+\sigma_1^2\right)a_1^2-\frac{f_1^2}{4\omega_1^2}=0 \qquad (20)$$

Generally, it has only one acceptable solution a_1 which increases almost linearly with f_1, but for some parameters the number of solutions is three.

4. Numerical results

In this section, our aim is to numerically validate the approximate solutions obtained in the previous part. Thus, Fig. 1 presents a classical variation of amplitudes $a_n, n=1,2$, with time, as calculated by integrating equations (16). Some oscillations are present initially but, as time increases, the curves approach horizontal lines corresponding to steady-state motions. The amplitudes becomes increasingly larger as the wave frequency tends to roll frequency.

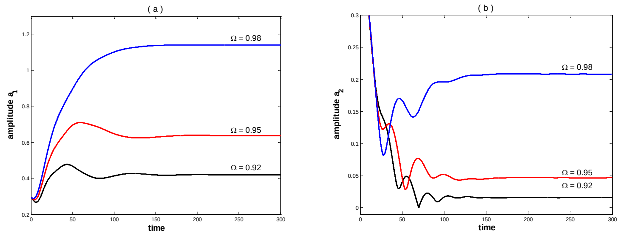
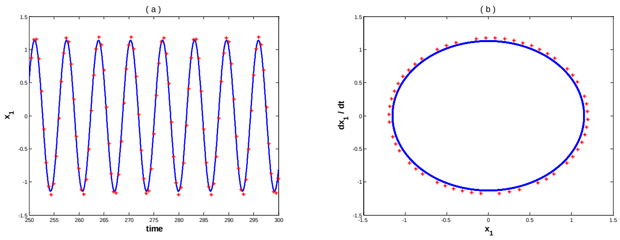


Fig. 1. *Dependence of amplitudes* a_1 *and* a_2 *on time t for* $\mu_1=0.25$, $\mu_2=0.5$, $\alpha_1=1, \alpha_2=2, \omega_1=1, \omega_2=2.1, F_1=0.07, F_2=0.05$, $\varepsilon=0.1$, *and different excitation frequencies* Ω.

With amplitudes $a_n, n=1,2$ and phases $\varphi_n, n=1,2$, provided by (16), one can use (18) to represent the first-order approximations for the solutions of (1). Figs. 2 and 3 show both the transient and long-term behaviors and phase planes for rolling and pitching, respectively.



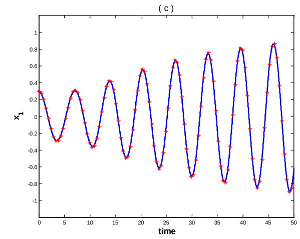


Fig. 2. *Time history and phase plane for roll mode with* $\Omega=0.98$. *The other parameters are the same as in Figure 1. Red asterisks stand for the numerical solution while blue curves represent approximate solution (18).*

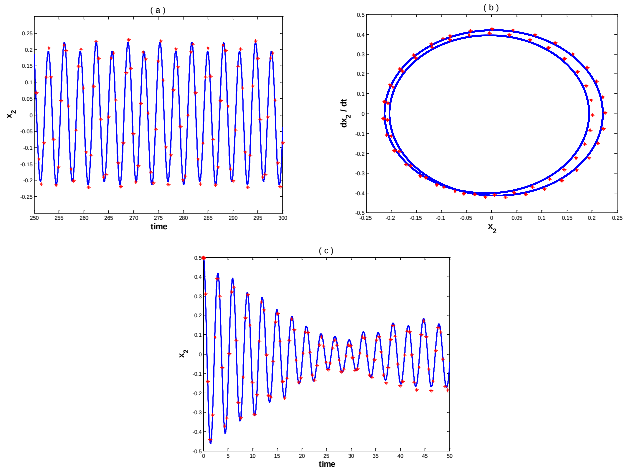


Fig. 3. *Time history and phase plane for pitch mode with* $\Omega=0.98$. *The other parameters are the same as in Figure 1. Red asterisks stand for the numerical solution while blue curves represent approximate solution (18).*

It is obvious that the analytical solution (18) given by Multiple Scales method is in a pretty good agreement with the numerical one. Then, we have contrasted the amplitudes of $x_n, n=1,2$, given by (19) and (20) with those obtained by numerical integration of (1). For the rest of the paper, the range for the external frequency Ω was thought to be [0.9, 1.1]. In Figure 4, all parameters are selected within the preordered range and different ratios of natural frequencies are considered. As expected, the graphs provided by both methods are practically indistinguishable, so only numerical frequency-response curves are displayed.

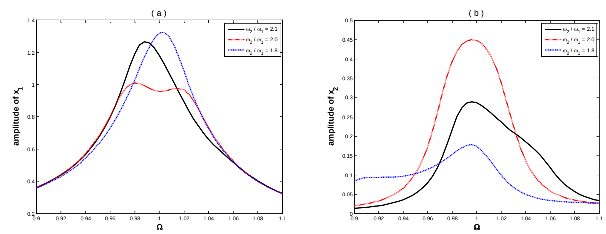


Fig. 4. *Amplitudes of roll and pitch modes versus excitation frequency for different ratios* ω_2/ω_1. *The other parameters are the same as in Figure 1.*

In the last part of the section, the effects of changing the system's parameters are investigated for the case $\omega_2/\omega_1=2$ only. Thus, Fig. 5 shows that for large damping the frequency – response curve has only one maximum, while for small damping the two maxima and one minimum suggested in Fig. 4 are better highlighted. Multiple Scales method results are labeled by red asterisks while the results provided by numerical integration are associated to dark continuous curves.

Fig. 6 reports a similar trend with decreasing / increasing of the external excitation amplitude F_1. F_2's modification has no practically effect on amplitude a_1, for a fixed F_1.

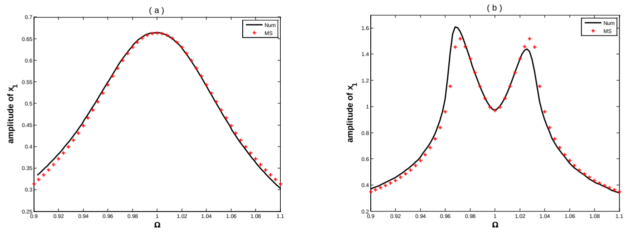


Fig. 5. *Amplitude of roll mode versus excitation frequency for* $\omega_2/\omega_1 = 2$.
a) $\mu_1 = 0.5, \mu_2 = 1$; *b)* $\mu_1 = 0.125, \mu_2 = 0.25$.
The other parameters are the same as in Figure 1.

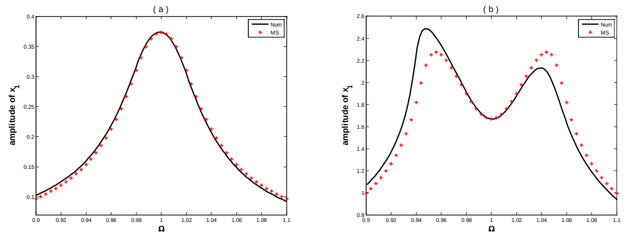


Fig. 6. *Amplitude of roll mode versus excitation frequency for* $\omega_2/\omega_1 = 2$.
a) $F_1 = 0.02$; *b)* $F_1 = 0.2$.
The other parameters are the same as in Figure 1.

Finally, Fig. 7 describes a mixed situation in which the nonlinearities were increased while the damping was decreased. For small forcing F_1 the only obvious change with respect to Figs. 5b and 6b consists in a clearer separation of extrema. For an increased value of F_1 there exist Ω values for that equation (20) has three positive solutions while numerical investigation shows a strange and totally different behavior in the neighborhood of the primary resonance $\Omega \approx \omega_1$ and two jumps from low to high amplitudes near $\Omega \approx 0.9$ and conversely for $\Omega \approx 1.07$.

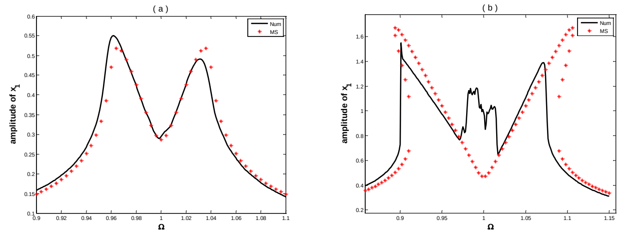


Fig. 7. *Amplitude of roll mode versus excitation frequency for* $\omega_2/\omega_1 = 2$, $\mu_1 = \mu_2 = 0.2, \alpha_1 = \alpha_2 = 5$, *and a)* $F_1 = 0.3$; *b)* $F_1 = 1$.
The other parameters are the same as in Figure 1.

5. Conclusions

In the paper, the nonlinear responses of a two-degrees-of-freedom model for pitch and roll ship motions have been studied both analytically and numerically for the particular case in which the pitch frequency is almost twice the roll frequency. The differential equations of motion are weakly nonlinear, thus we have searched, by means of a straightforward expansion of the solution, the resonant values of the external excitation frequency. Between the obtained resonant frequencies we have selected a primary one, namely that where the encounter frequency is close to the roll frequency. The Multiple Scales method permitted us to derive the governing equations for the transition towards the steady-state solutions, the first-order approximations for these solutions and the frequency-amplitudes relationships. The reliability of the analytical results derived in the paper was checked by comparing them with the numerical solutions provided by an ODEs integrator. Using time series plots and frequency-amplitude curves, we have proved that the two solutions match well if the system's parameters were selected without order violations.

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