DEVELOPMENT OF A CANTILEVER PLATE'S MODEL AS A COMPONENT OF THE METHOD OF LOCAL APPROXIMATIONS

РАЗВИТИЕ МОДЕЛИ КОНСОЛЬНОЙ ПЛАСТИНЫ КАК КОМПОНЕНТА МЕТОДА ЛОКАЛЬНЫХ АППРОКСИМАЦИЙ

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Abstract: *The paper contains a numerical-analytical solution for deflections and force factors in a cantilever plate of infinite length due to a concentrated load, which is obtained using modern data about optimal quadrature integration. The aspect of optimality is the main feature that distinguished the present solution. The formulas provide high accuracy, which is examined through comparison of the results with the finite elements method and another numerical-analytical solution. The paper also contains formulas for deflections for different laws of load distribution and ideas for their application.*

KEYWORDS: *THEORY OF ELASTICITY, STRESSED STATE, THE METHOD OF LOCAL APPROXIMATIONS, KIRCHHOFF PLATE, DEFLECTIONS, GEAR MODELING*

1. Introduction

Due to rapid development of technologies and materials usable in mechanical engineering, the need of strength calculations' clarification rises for innovative purposes. In this context, theoretical studies of stress concentration in complex-shaped bodies with loaded ledges (such as gearings, for example), which are based on separate analysis of force factors due to an applied load, are extremely valuable. As it is noticed in [1], the standardized methods for calculating numerous sets of details of different mechanisms and machines with loaded ledges (e.g., the Russian GOST - State Standard №21354-87 "Cylindrical Involute Gear Transmissions: Strength Calculations", 1987) for bending strength of the ledges are guided by simplified conceptions. They provide neither information about force factors' distribution laws nor ways for modeling different materials with complex properties. On the other hand, there is a very effective tool – the finite elements method, which has high prevalence among engineers and researchers. However, it demands much of constructing the model properly, and has some specific disadvantages such as the locking effect, for example, which can give troubles if you desire to compute the result with high accuracy. Other reasons for building analytical solutions in parallel with improvement of the FEM modeling are, of course, urge towards having comparative results and general regularities, which could be made more precise with the FEM.

The present paper contains some basic ideas about the method of local approximations (the MLA) – a method of investigation of the stressed state of complex-shaped bodies with loaded ledges. G. A. Zhuravlev proposed it in [2, 3] in the context of modernization of a gearing's geometry accordingly to the actual stress concentration, which takes into account origin of every stress component. The method is based on two plane problems, to which the three-dimensional one is being reduced. Thus, we calculate the force factors on a cantilever plate (using classic or improved theories of plate bending) in order to use them in calculation of the respective stress components in the model of a rod with two deep symmetric hyperbolic recesses. In this connection we consider, that the contour of the hyperbolic recess approximates the contour of the ledge, and take into account the depth of the plate's sealing. It is shown in the paper [4], how the first requirement can be satisfied for a tooth of a spur pinion. The question about the depth of the sealing is observed, for example, in [3].

The problem of stress concentration in a rod with two deep symmetric hyperbolic recesses is solved in [5]. As for the problem of a cantilever plate calculation, the number of papers, which contain analytical or numerical-analytical solutions, is not big. The present paper describes a comparatively simple solution, which corresponds to a model of an infinitely long plate, clamped along one of its long sides. T. J. Jaramillo described this problem in [6] and gave an exact solution in terms of improper integrals for the deflections and moments due to a transverse concentrated load acting at an arbitrary point of the plate. It was proposed in [6] to calculate the respective integrals using residue theory, but as it was shown in the papers [7, 8] and in the Proceedings of 2015 International Conference on "Physics and Mechanics of New Materials and Their Applications" (PHENMA 2015), the residue solution of [6] gives a big error for some calculation points. The present paper continues the course of [7] and presents a numerical-analytical solution for deflections (which is basic for calculating force factors and stresses), built with applying modern ideas about optimal quadrature integration.

We also introduce formulas for deflections, suitable for different laws of load distribution, such as uniformly distributed over a segment and over a rectangle loads, a distributed over a segment accordingly to the parabolic law load and a load, distributed over an ellipse.

2. Deflections

2.1. A numerical-analytical solution for deflections

On the analogy of [6], the plate is considered in the form of an infinitely long strip of width A and thickness h, fixed along one of its long edges. The transverse load F is applied at an arbitrary point $P(c,0)$ of the plate, where the Cartesian coordinate system $Oxyz$ is located such that xy is the middle plane of the plate, Oy belongs to the fixed side and the line of F belongs to the xz plane. Let the plane $x=c$ divide the plate and the functions, which are determined in the areas $0 \le x \le c$ and $c \le x \le A$, have the indexes 1 and 2 respectively. Then, if the weight of the plate isn't considered the deflections $W_j(x,y)$, $j=1,2$, satisfy the biharmonic equation

$$\nabla^4 W_j(x,y)=0, \quad j=1,2,$$

which is valid everywhere except the point P.

It is proposed in [7] to define the functions $W_j(x,y)$, $j=1,2$, as

$$W_j(x,y)=\int_0^\infty f_j(x,\alpha)\cos(\alpha y)d\alpha,$$

where

$$f_j(x,\alpha)=(A_j+B_j\alpha x)\cosh(\alpha x)+(C_j+D_j\alpha x)\sinh(\alpha x),$$

A_j, B_j, C_j, D_j are assumed functions of the α variable.

Then the boundary conditions can be written as

$$\left.\begin{array}{l} f_1(0,\alpha)=f_{1,x}(0,\alpha)=0;\\ f_{2,xx}(A,\alpha)-\alpha^2\mu f_2(A,\alpha)=0;\\ f_{2,xxx}(A,\alpha)-\alpha^2(2-\mu)f_{2,x}(A,\alpha)=0;\\ f_1(c,\alpha)-f_2(c,\alpha)=0;\\ f_{1,x}(c,\alpha)-f_{2,x}(c,\alpha)=0;\\ f_{1,xx}(c,\alpha)-f_{2,xx}(c,\alpha)=0;\\ f_{2,xxx}(c,\alpha)-f_{1,xxx}(c,\alpha)=\dfrac{F}{\pi D}, \end{array}\right\}$$

where $D=\dfrac{Eh^3}{12(1-\mu^2)}$ is the flexural rigidity of the plate (E is Young's modulus, μ is Poisson's ratio).

Being solved for A_j, B_j, C_j, D_j, $j=1,2$, the system provides the following functions:

$$f_1(x,c,\alpha)=\frac{F}{2\pi AD\gamma}\left[-x\alpha\cosh(x\alpha)\sum_{i=1}^{4}a_ik_i+\sinh(x\alpha)\sum_{i=4}^{8}a_ik_i\right],$$

$$f_2(x,c,\alpha)=f_1(c,x,\alpha),$$

$$\gamma=\alpha^3\left[5+2(A\alpha)^2(\mu-1)^2+(2+\mu)\mu-(\mu^2+2\mu-3)\cosh(2A\alpha)\right]$$

i	k_i	a_i
1	$5+2A(A-c)\alpha^2(\mu-1)^2+(2+\mu)\mu$	$\cosh(c\alpha)$
2	$-(\mu^2+2\mu-3)$	$\cosh[(c-2A)\alpha]$
3	$\alpha(\mu-1)(2A-c)(\mu-1)$	$\sinh(c\alpha)$
4	$\alpha c(\mu-1)(\mu+3)$	$\sinh[(c-2A)\alpha]$
5	$5+(2A^2-2Ac+cx)\alpha^2(\mu-1)^2+(2+\mu)\mu$	$\cosh(c\alpha)$
6	$-(1+cx\alpha^2)(\mu^2+2\mu-3)$	$\cos[(c-2A)\alpha]$
7	$\alpha\begin{bmatrix}-c(\mu-1)^2+2A^2x\alpha^2(\mu-1)^2-\\ -2A(cx\alpha^2-1)(\mu-1)^2+4x(\mu+1)\end{bmatrix}$	$\sinh(c\alpha)$
8	$c\alpha(\mu^2+2\mu-3)$	$\sinh[(c-2A)\alpha]$

Thus,

$$W_j(x,y,c)=\int_0^\infty f_j(x,c,\alpha)\cos(y\alpha)\,d\alpha,\ j=1,2,$$

and for numerical calculation of the deflections we have proposed in [10] the following formula:

$$W_j(x,y,c)=\frac{\pi}{2n}\sum_{k=1}^{n}F_j\left[x,y,c,\tan\frac{(2k-1)\pi}{4n}\right],\ j=1,2 \qquad (1)$$

where

$$F_j(x,y,c,\alpha):=(1-\alpha^2)f_j(x,c,\alpha)\cos(\alpha y),\ j=1,2,$$

and the error of (1) is $\dfrac{\pi}{4n}$.

Due to the behavior of the respective functions it is proposed to use different values of n for $j=1$ and for $j=2$. The main feature of the formula (1) is that it can be called an almost optimal one, because we use the respective result from theory of optimal quadrature integration in order to build it. However, some additional assumptions, which take into account the behavior of the integrands and substantiate selection of n values, have to be made. It is convenient to use dimensionless coordinates

$$\xi=\frac{x}{A},\ \eta=\frac{y}{A},\ \zeta=\frac{c}{A}.$$

Computational experiments show that the values of deviation between the results of the FEM modeling and the formula (1) become more than 1% if η approach 1,00. Here we consider the following parameters of the model in ANSYS:

Length of the plate L, m	h, m	A, m	F, N	Finite elements type	Multiplicity of splitting: the side of L, the side of h, the side of A
0,2	0,002	0,01	2000	SHELL63	400
					4
					40

It is obvious that the results directly depend on the ratios of $\dfrac{L}{A}$ and $\dfrac{y}{L}$ in the FEM model; thus, we have to consider a sufficiently long plate and calculation points, which are sufficiently removed from the side edges of the plate.

2.2. Different laws of load distribution

The following formulas in terms of improper integrals correspond to different cases of a distributed load, such as uniformly distributed over a segment and over a rectangle loads, a distributed over a segment accordingly to the parabolic law load and a load, distributed over an ellipse. Hereinafter we suppose, that r is a ratio of a distance between the sealing line of the plate and the center of a loaded area to the size A, F_Σ is a distributed load, q_0 is intensity of loading in the center point of a loaded area and $\varphi_j(\xi,\zeta,\mu)=\dfrac{f_j(\xi,\zeta,\mu)}{F}\mu^3\kappa(\mu)$, $j=1,2$, where $\mu=\alpha A$, $\kappa(\mu)=\mu^2+\theta^2+(2\theta+1)\cosh^2(\mu)$, $\theta=\dfrac{1+\nu}{1-\nu}$. The following formulas are proposed on the base of [8].

If the load is uniformly distributed over an interval of length 2ρ, which is parallel to the sealing line, than

$$W_j(\xi,\eta,\zeta)=\frac{F_\Sigma}{q_0}\int_0^\infty\frac{\varphi_j(\xi,r,\mu)}{\mu^4\kappa(\mu)}\cos(\mu\eta)\sin(\mu\eta)\,d\mu.$$

If the load is distributed over the same segment, divided into k elementary segments of equal lengths ΔS, accordingly to the parabolic law

$$q=\frac{3F_\Sigma(\rho^2-\varepsilon^2)}{4\rho^3},$$

where ε is the distance between the center of the segment and the calculation point, then

$$W_j(\xi,\eta,\zeta)=\frac{3F_\Sigma}{q_o^3}\int_0^\infty\frac{\varphi_j(\xi,r,\mu)}{\mu^6\kappa(\mu)}\cos(\mu\eta)\left[\sin(\mu\rho)-\mu\rho\cos(\mu\rho)\right]d\mu.$$

If a load of intensity q is uniformly distributed over a rectangle $\zeta_1\le\xi\le\zeta_2$, $\eta_1\le\eta\le\eta_2$, then

$$W_j(\xi,\eta,\zeta)=4q\int_0^\infty\frac{\cos[(\eta-\bar\eta)\mu]\sin(\mu\nu)}{\mu^5\kappa(\mu)}\bar\varphi(\xi,\mu)\,d\mu,$$

where

$$\bar\varphi(\xi,\mu)=\sum_{j=1}^{2}\left\{\begin{array}{l}\sinh(\mu u_j)\varphi_j(\xi,\bar\xi_j,\mu)+\\ \quad+\left[\mu u_j\cosh(\mu u_j)-\sinh(\mu u_j)\bar\varphi_j(\xi,\bar\xi,\mu)\right]\end{array}\right\},$$

$$\bar\varphi_j(\xi,\zeta,\mu)=(1+2\theta)(2\mu\xi C_2+S_2-S_1)-2\mu(1-\xi)C_4-S_4+2\mu C_5+t_jS_5,$$

$$t_1=1+4\mu^2\xi,\ t_2=(1+2\theta)^2+4\mu^2(1-\xi),$$

$$S_1=\sinh[(2-\zeta+\xi)\mu],\qquad S_2=\sinh[(2-\zeta-\xi)\mu],$$

$$S_4=\sinh[(\zeta+\xi)\mu],\qquad S_5=\sinh[(\zeta-\xi)\mu],$$

$C_2 = \cosh\left[(2-\zeta-\xi)\mu\right]$, $C_4 = \cosh\left[(\xi+\zeta)\mu\right]$,

$C_5 = \cosh\left[(\zeta-\xi)\mu\right]$, $v = \frac{\eta_2-\eta_1}{2}$, $\bar{\eta} = \frac{\eta_1+\eta_2}{2}$,

$$u_1 = \begin{cases} \frac{1}{2}\left[\zeta_2 - \max(\xi,\zeta_1)\right], & \xi \le \zeta_2; \\ 0, & \xi > \zeta_2, \end{cases}$$

$$u_1 = \begin{cases} \frac{1}{2}\left[\min(\xi,\zeta_2) - \zeta_1\right], & \xi \ge \zeta_1; \\ 0, & \xi < \zeta_1, \end{cases}$$

$$\bar{\xi}_1 = \frac{1}{2}\left[\zeta_2 + \max(\xi,\zeta_1)\right],\ \bar{\xi}_2 = \frac{1}{2}\left[\min(\xi,\zeta_2) + \zeta_1\right].$$

At last, if the load is applied accordingly to the law of semi-ellipsoid over an ellipse with dimensionless semi-axes a and b, which is inclined by an angle Λ, it is proposed in [8] to use a method, based on reduction of the problem to the case of the parabolic law of load distribution. Thus, we divide the ellipse into elements with a system of equidistant chords, parallel to the free edge. The intensity of the load inside every element is

$$q(\varepsilon,\zeta) = q_0 \left\{ 1 - \frac{\left[\varepsilon\cos(\Lambda) + (\zeta - r)\sin(\alpha)\right]^2}{a^2} - \frac{\left[\varepsilon\sin(\Lambda) - (\zeta - r)\cos(\alpha)\right]^2}{b^2} \right\}^{\frac{1}{2}},$$

where $q_0 = \frac{3}{2}\frac{F_\Sigma}{\pi ab}$, and we replace it with an equivalent one, which is distributed accordingly to the parabolic law over the chord, which contains the application point of the resultant force of the element. We introduce the following notation:

NN' is the diameter of the ellipse, connected with the chosen system of dividing chords; $S_0 = \frac{NN'}{2}$; S is a linear coordinate, which is counted off the center of the ellipse along the diameter NN'; n_e is a number of parts, into which the ellipse is divided; λ_i, $i = 1,2,...,n+1$, are the coordinates of points of dividing chords' intersection with NN'; F_k, $k = 1,...,n$, is the value of the resultant force on the element between λ_k and λ_{k+1}; l_k is the coordinate of the point of the resultant force acting; ρ_k is one-half of the chord, along which the equivalent load is distributed; β is the angle between NN' and $O\xi$.

The following relations link all of the introduced parameters:

$$\lambda_0 = \sqrt{\frac{a^4\sin^4(\Lambda) + b^4\cos^2(\Lambda)}{a^2\sin^2(\Lambda) + b^2\cos^2(\Lambda)}};$$

$$\lambda_i = -\lambda_0 + (i-1)\frac{2\lambda_0}{n_e},\ i = 1,2,...,n_e+1;$$

$$F_k = \frac{\pi abq_0}{3n_e\lambda_0^2}\left[3\lambda_0^2 - \left(\lambda_{k+1}^2 + \lambda_k\lambda_{k+1} + \lambda_k^2\right)\right],\ k = 1,2,...,n_e;$$

$$l_k = -\frac{Aabq_0}{4\pi n F_k \lambda_0^2}\left(\lambda_{k+1} + \lambda_k\right)\left(\lambda_{k+1}^2 + \lambda_k^2 - 2\lambda_0^2\right),$$

$$\rho_k = \frac{ab}{\lambda_0}\sqrt{\frac{\lambda_0^2 - l_k^2}{a^2\sin^2(\Lambda) + b^2\cos^2(\Lambda)}},$$

$$\cos(\beta) = \frac{a^2\sin^2(\Lambda) + b^2\cos^2(\Lambda)}{\sqrt{a^4\sin^2(\Lambda) + b^4\cos^2(\Lambda)}}.$$

Thus, the method provides the following:

$$W_j = 3\sum_{k=1}^{n}\frac{F_k}{\rho_k^3}\int_0^\infty \frac{\varphi_j\left(\xi, r - l_k\cos(\beta), \mu\right)}{\mu^6\kappa(\mu)}\cos\left[\left(\eta + l_k\sin(\beta)\right)\mu\right]\cdot \cdot\left[\sin(\mu\rho_k) - \mu\rho_k\cos(\mu\rho_k)\right]d\mu.$$

2.3. Practical recommendations for calculation

The problem of the deflections' calculation for the observed cases of load distribution consists in calculation of the respective improper integrals, which contain complex integrands. We suppose that the most effective way is building highly accurate approximate formulas, which take into account not only behavior of the integrands, but also correctly reflect the influence of the respective derivatives. The latter is very important for creation calculation algorithms for force factors and stresses.

Some approximate formulas can be found in [8]. Though they include strict limitations for calculation points and are not universal concerning material properties, they provide satisfying accuracy for steel plates within the limits of their application.

It is also useful to notice, that for deflections at the points located in the area of loading it is not hard to reduce the problem to consecutive repeated application of the formula (1). This is shown in the paper [1], where an algorithm for the case of the parabolic law of load distribution is represented.

3. Force factors due to a concentrated load

In the proceedings of PHENMA 2015 we have proposed the following formulas for the force factors due to a concentrated load – respectively, shearing force, bending and twisting moments:

$$Q_{jx}(x,y,c) = -\frac{\pi D}{2n}\sum_{k=1}^{n} F_j^{Q_x}\left[x,y,c,\tan\frac{(2k-1)\pi}{4n}\right],$$

$$M_{jx}(x,y,c) = -\frac{\pi D}{2n}\sum_{k=1}^{n} F_j^{M_x}\left[x,y,c,\tan\frac{(2k-1)\pi}{4n}\right],$$

$$M_{jxy}(x,y,c) = -\frac{\pi D(1-\mu)}{2n}\sum_{k=1}^{n} F_j^{M_{xy}}\left[x,y,c,\tan\frac{(2k-1)\pi}{4n}\right],$$

where

$$F_j^{Q_x}(x,y,c,\alpha) = \left\{ \begin{array}{l} \left[f_j(x,c,\alpha)\cos(\alpha y)\right]_{xxx} + \\ +\left[f_j(x,c,\alpha)\cos(\alpha y)\right]_{xyy} \end{array} \right\}\left(1-\alpha^2\right),$$

$$F_j^{M_x}(x,y,c,\alpha) = \left\{ \begin{array}{l} \left[f_j(x,c,\alpha)\cos(\alpha y)\right]_{xx} + \\ +\mu\left[f_j(x,c,\alpha)\cos(\alpha y)\right]_{yy} \end{array} \right\}\left(1-\alpha^2\right),$$

$$F_j^{Q_x}(x,y,c,\alpha) = \left[f_j(x,c,\alpha)\cos(\alpha y)\right]_{xy}\left(1-\alpha^2\right).$$

This solution was compared with one, which could be found in [9] for a steel plate. It is shown, that the results of [9] are qualitatively improved.

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