THE NATURAL FREQUENCIES AND MODE SHAPES OF AN EULER-BERNOULLI BEAM WITH A RECTANGULAR CROSS- SECTION WHICH HAS A SURFACE CRACK

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Abstract: *The natural frequencies and mode shapes of an Euler-Bernoulli beam with a rectangular cross- section, which has a surface crack, is investigated. The crack is modeled as a change (sudden or gradual) in the cross-section of the beam, and the perturbation approach is used assuming that the crack is much smaller than the beam cross section. Computations of natural frequencies and mode shapes were carried out for four different crack shapes with rectangular, triangular and parabolic profiles when viewed through the side of the beam. The results are listed in non-dimensional form for various values of the parameters characterizing the crack.*

Keywords: EULER BERNOULLI, BEAM, CRACK

1. Introduction

Identification and characterization of cracks in engineering structures is an important problem both theoretically and technically. One type of problem deals with determination of vibration characteristics of the structure with a crack. Another problem might be the characterization of the crack from measured vibration characteristics of the structure; this is usually called an inverse problem. The practical solution of the inverse problem normally involves solving the forward problem for a wide range of crack types. Therefore, the ability to solve the vibration characteristics of a cracked structure is very important.

Most studies dealing with cracked structures in the literature models the crack as a change in the elastic characteristics of the structure, and utilize some type of numerical method. The present study aims to model the detailed geometry of the crack therefore shedding light on the solution of the inverse problem. Furthermore the perturbation theory approach will be used making the solution analytical.

One of the earlier studies, Adams et al. [1], demonstrated a vibration method for non-destructive testing of structures. Shen and Pier [2] studied the convergence of Galerkin approach for beams with symmetric cracks. Papaeconomou and Dimarogonas [3] describe a transfer matrix model for a cracked prismatic bar. Chondros and Dimarogonas [4] used a variational formulation to analyze lumped and continuous cracks. Khiem and Toan [5] used a modification of the Rayleigh quotient method for detection of an unknown number of multiple cracks on beams. Saez et al. [6] performed damage detection by solving the inverse problem. Chaudhari and Maiti [7] used a rotational spring to represent the crack and worked on solving the inverse problem based on the measurement of natural frequencies. He and Lin [8] uses an acoustic system for contact-type cracks. Nejad et al. [9] worked on analytical estimation of natural frequencies and mode shapes of a beam having two cracks. Mazanoglu et al. [10] modified the energy-based method presented by Yang et al. [11], to solve the vibration of non-uniform Euler–Bernoulli beams with multiple cracks by defining the crack as a spring. Open edge cracks were investigated by Aydin [12] again modeling the crack as a spring. Caddemi and Morassi [13] proposed a justification of the rotational elastic spring model of an open crack in a beam. Finite element method is very popular for investigation of free vibration analysis of cracked beams [14-17]. Cracks in reinforced concrete structures is another popular research topic [18,19] using Euler-Bernoulli Beam theory. Timoshenko Beam model was also used [20, 21].

The studies mentioned above do not consider the shape of the crack, but model it as a spring or a change in the local elastic properties. In this study the crack is expressed as a change in the beam cross-section as explained in Section 2. This allows the determination of vibration characteristics depending on the shape of the crack.

2. Perturbation Method and Governing Equations

Considering an Euler-Bernoulli beam of length L and cross-sectional area A(x), the governing equation for the vibrations is

$$-\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y}{\partial x^2}\right) = m(x)\frac{\partial^2 y}{\partial t^2} \tag{1}$$

where the cross-sectional area of the beam, $A(x)$, varies with coordinate x along the beam. $y(x,t)$ is the displacement, E is the elastic modulus of the beam material, $I(x)$ is the second area-moment of cross section, $m(x)$ is the mass per unit length. The cross section is assumed to be a rectangle with width $b(x)$, and height $h(x)$ changing along the beam; thus

$$A(x) = b(x)h(x) \tag{2a}$$

$$I(x) = \frac{1}{12}b(x)h(x)^3 \tag{2b}$$

$$m(x) = \rho b(x)h(x) \tag{2c}$$

where ρ is the density of the beam material. Both ends of the beam are assumed to be simply-supported,

$$y = \frac{\partial^2 y}{\partial x^2} = 0 \text{ at } x = 0, L \tag{3}$$

The non-damaged beam has uniform cross section with constant width and height b_0 and h_0 respectively. The crack is modeled as a change in the beam cross-sectional dimensions in the form

$$b(x) = b_0 + \varepsilon f(x) \tag{4a}$$

$$h(x) = h_0 + \varepsilon g(x) \tag{4b}$$

where ε is a small non-dimensional perturbation parameter and the functions $f(x)$ and $g(x)$ determine the shape of the crack; these are completely general at this point. Substituting (4) into (2) and (1), the vibration equation becomes, ignoring higher order terms,

$$-\frac{\partial^2}{\partial x^2}\left(\frac{E}{12}(b_0h_0^3 + \varepsilon(3b_0h_0^2g(x) + h_0^3f(x)))\frac{\partial^2 y}{\partial x^2}\right) = \rho(b_0h_0 + \varepsilon(b_0g(x) + h_0f(x))\frac{\partial^2 y}{\partial t^2} \tag{5}$$

The governing equation is non-dimensionalized with the following definitions, starred symbols showing non-dimensional variables,

$$x^* = \frac{x}{L} \tag{6a}$$

$$t^* = \frac{t}{\sqrt{\rho \frac{12L^4}{Eh_0{}^4}}} \tag{6b}$$

$$y^* = \frac{y}{h_0} \tag{6c}$$

$$G^*(x^*) = \frac{g(x)}{h_0} \tag{6d}$$

$$F^*(x^*) = \frac{f(x)}{b_0} \tag{6e}$$

The non-dimensional vibration equation becomes, omitting stars after this point,

$$\frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = -\varepsilon \left[\begin{array}{c} \frac{\partial^2}{\partial x^2}\left(\begin{array}{c} (3G(x) \\ +(x))\frac{\partial^2 y}{\partial x^2} \end{array} \right) \\ +(G(x)+F(x))\frac{\partial^2 y}{\partial t^2} \end{array} \right] \tag{7}$$

and the boundary conditions

$$y = \frac{\partial^2 y}{\partial x^2} = 0 \text{ at } x = 0,1 \tag{8}$$

Assuming a separated solution of the form

$$y(x,t) = u(x)p(x) \tag{9}$$

leads to

$$\ddot{p} + \lambda^2 p = 0 \tag{10a}$$

$$u'''' - \lambda^2 u + \varepsilon\varphi(u,\lambda) = 0 \tag{10b}$$

Prime and dot denote differentiation with respect to x and t, respectively, and we defined

$$\varphi(u,\lambda) = ((3G(x)+F(x))u'')'' - \lambda^2 u(G(x)+F(x)) \tag{11}$$

Solution of Eq. (10b) is assumed to be in the form of a perturbation series for both the mode shape and the eigenvalue

$$u(x) = u_0(x) + \varepsilon u_1(x) + \ldots \tag{12a}$$

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \ldots \tag{12b}$$

Substituting, the zero and first order problems become

$$u_0'''' - \lambda_0{}^2 u_0 = 0 \tag{13a}$$

$$u_0(0) = u_0''(0) = u_0(1) = u_0''(1) = 0 \tag{13b}$$

and

$$u_1'''' - \lambda_0{}^2 u_1 - 2\lambda_0\lambda_1 u_0 + \varphi(u_0,\lambda_0) = 0 \tag{14a}$$

$$u_1(0) = u_1''(0) = u_1(1) = u_1''(1) = 0 \tag{14b}$$

The solution of the zero order problem is

$$u_0 = \sin n\pi x,\ \lambda_0 = n^2\pi^2,\ n = 1,2,3,\ldots \tag{15}$$

Thus, the first order problem becomes

$$\begin{aligned} u_1'''' - n^4\pi^4 u_1 &= 2n^2\pi^2\lambda_1 \sin n\pi x \\ &-\varphi(\sin n\pi x, n^2\pi^2) \end{aligned} \tag{16}$$

The solution of (16) can be written by variation of constants as

$$\begin{aligned} u_1 &= C_1(x)e^{n\pi x} + C_2(x)e^{-n\pi x} \\ &+C_3(x)\cos n\pi x + C_4(x)\sin n\pi x \end{aligned} \tag{17}$$

Where

$$C_1'(x) = \frac{e^{-n\pi x}(2n^2\pi^2\lambda_1 \sin n\pi x + \varphi(\sin n\pi x, n^2\pi^2))}{4n^3\pi^3} \tag{18a}$$

$$C_2'(x) = \frac{e^{n\pi x}(2n^2\pi^2\lambda_1 \sin n\pi x + \varphi(\sin n\pi x, n^2\pi^2))}{4n^3\pi^3} \tag{18b}$$

$$C_3'(x) = \frac{\sin n\pi x(2n^2\pi^2\lambda_1 \sin n\pi x + \varphi(\sin n\pi x, n^2\pi^2))}{2n^3\pi^3} \tag{18c}$$

$$C_3'(x) = -\frac{\cos n\pi x(2n^2\pi^2\lambda_1 \sin n\pi x + \varphi(\sin n\pi x, n^2\pi^2))}{2n^3\pi^3} \tag{18d}$$

Integrating (18) and substituting in (17) gives the general solution in the form

$$\begin{aligned} u_1 &= K_1 e^{n\pi x} + K_2 e^{-n\pi x} + K_3 \cos n\pi x + K_4 \sin n\pi x \\ &+\int_0^x \left[\begin{array}{c} C_1(\xi)e^{n\pi\xi} + C_2(\xi)e^{-n\pi\xi} \\ +C_3(\xi)\cos n\pi\xi + C_4(\xi)\sin n\pi\xi \end{array} \right] \Psi(\xi)d\xi \end{aligned} \tag{19}$$

where we defined

$$\Psi(x) = n^2\pi^2\lambda_1 \sin n\pi x - \varphi(\sin n\pi x, n^2\pi^2) \tag{20}$$

for brevity. *K*'s are arbitrary constants found by applying the boundary conditions (14b); the result is the system of equations

$$K_1 + K_2 + K_3 = 0 \tag{21a}$$

$$K_1 + K_2 - K_3 + \frac{1}{2n^4\pi^4}\left[(-1)^n - \frac{1}{n\pi}\right]\psi(0) = 0 \tag{21b}$$

$$\begin{aligned} &K_1 e^{n\pi} + K_2 e^{-n\pi} + K_3(-1)^n + \frac{e^{n\pi}}{4n^3\pi^3}\int_0^1 e^{n\pi\xi}\psi(\xi)d\xi \\ &-\frac{e^{-n\pi}}{4n^3\pi^3}\int_0^1 e^{n\pi\xi}\psi(\xi)d\xi + \frac{(-1)^n}{2n^3\pi^3}\int_0^1 \sin n\pi\xi\psi(\xi)d\xi = 0 \end{aligned} \tag{21c}$$

$$\begin{aligned} &K_1 e^{n\pi} + K_2 e^{-n\pi} - K_3(-1)^n + \frac{e^{n\pi}}{4n^3\pi^3}\int_0^1 e^{-n\pi\xi}\psi(\xi)d\xi \\ &-\frac{e^{-n\pi}}{4n^3\pi^3}\int_0^1 e^{n\pi\xi}\psi(\xi)d\xi - \frac{(-1)^n}{2n^3\pi^3}\int_0^1 \sin n\pi\xi\psi(\xi)d\xi = 0 \end{aligned} \tag{21d}$$

Note that K_4 does not show in these equations; this is due to the fact that the term associated with K_4 in the general solution, Eq. (19), is $\sin n\pi x$, and that is the solution of the zero order problem. This part, $K_4 \sin n\pi x$, is therefore discarded and seen as part of the zero order solution. The unknowns in Eqs.(21) are K_1, K_2, K_3 and λ_1 (inside ψ).

In evaluating the terms in Eqs.(21) we come across expressions like $F(0), F'(1), G(1)$, etc. If we assume that the crack is away from the ends ($x = 0$ *and* 1), all these terms become zero.

3. Conclusion

The results are listed in non-dimensional form for various values of the parameters characterizing the crack.

Table1: Crack Shape and λ

Crack Shape	$\lambda_1^{(1)}$	$\lambda_1^{(2)}$	$\lambda_1^{(3)}$	$\lambda_1^{(4)}$	$\lambda_1^{(5)}$	$\lambda_1^{(6)}$	$\lambda_1^{(7)}$	$\lambda_1^{(8)}$	$\lambda_1^{(9)}$	$\lambda_1^{(10)}$
Rectangular Crack	-1.95	-0.25	-16.50	-3.84	-40.38	-17.60	-66.15	-48.39	-88.68	-98.69
Triangle (Isosceles) Crack	-1.94	-0.49	-15.42	-6.74	-34.67	-26.48	-54.90	-59.71	-80.89	-98,69
Triangle (Right) Crack	-1.91	-0.96	-13.36	-12.09	-24.67	-41.07	-37.90	-75.11	-71.63	-98.69
Parabolic Crack	-1.95	-0.34	-16.11	-4.94	-38.10	-21.39	-60.89	-54.45	-83.19	-101.52

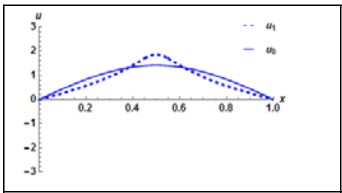
Fig.1 Rectangular crack first mode

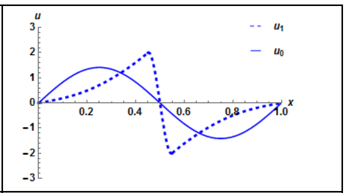
Fig.2 Rectangular crack second mode

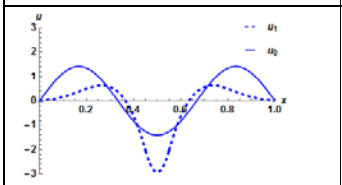
Fig.3 Rectangular crack third mode

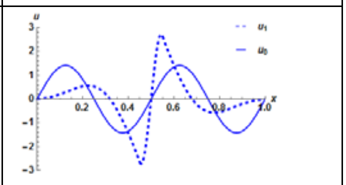
Fig.4 Rectangular crack fourth mode

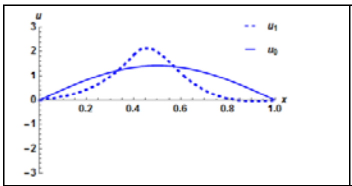
Fig.5 Triangular (right) crack first mode

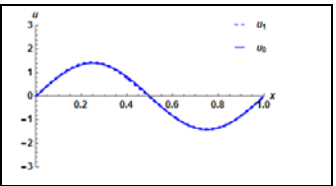
Fig.6 Triangular (right) crack second mode

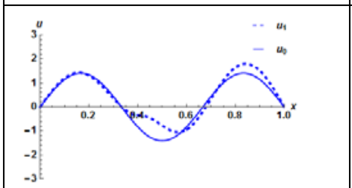
Fig.7 Triangular (right) crack third mode

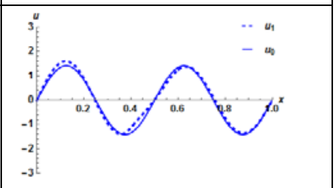
Fig.8 Triangular (right) crack fourth mode

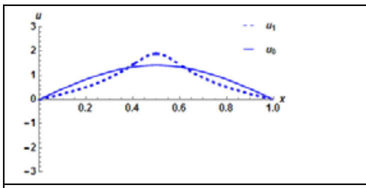
Fig.9 Triangular (isoscales) crack first mode

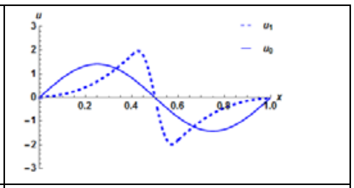
Fig.10 Triangular (isoscales) crack second mode

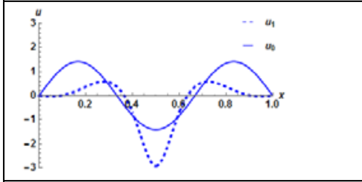
Fig.11 Triangular (isoscales) crack third mode

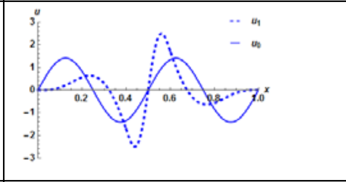
Fig.12 Triangular (isoscales) crack fourth mode

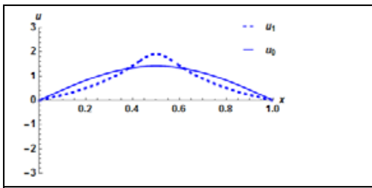
Fig.13 Parabolic crack first mode

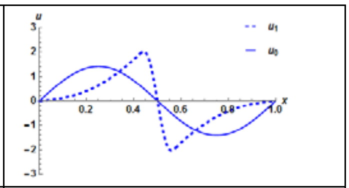
Fig.14 Parabolic crack second mode

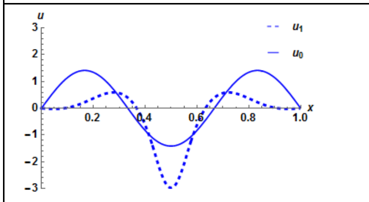
Fig.15 Parabolic crack third mode

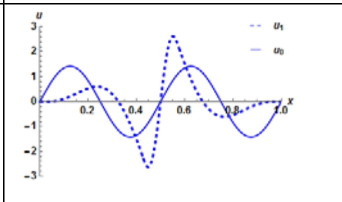
Fig.16 Parabolic crack fourth mode

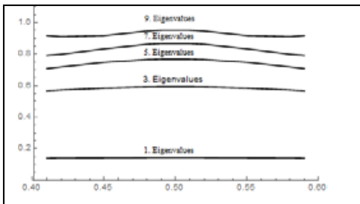
Fig.17 odd eigenvalues

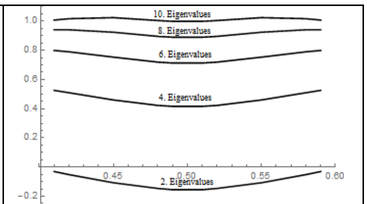
Fig.18 even eigenvalues

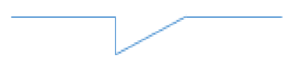
Fig.19 Triangular (right) crack


Fig.20 Triangular (isoscales)crack

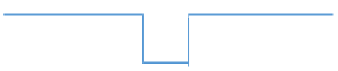
Fig.21 Rectangular crack


Fig.22 Parabolic crack

Nomenclature

A (x)	area of cross section
b (x), h (x)	width and height of beam cross section
b_0, h_0	width and height of non-cracked beam cross section
I (x)	second area moment
m (x)	mass per unit length
E	modulus of elasticity
ρ	density
L	length of the beam
ε	perturbation parameter
f (x), g (x)	crack shape functions
y (x, t)	vertical displacement
u (x)	mode shape
p (t)	separated time function
λ	eigenvalue
$[\ ]^*$	non-dimensional [ ]

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