**INDUSTRIAL MATHEMATICS – PHASE TRANSITIONS, SCATTERING, STRUCTURES**

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Abstract: Presents the basics of methodology to use the full knowledge of micro-producer (personality, micro-foundry) based on: methodology of mathematics, foundations of mathematics and mathematical physics. Example - phase transition of the first order of Stefan’s problems, scattering connection with new structures.

Key words: phase transition of firs order, Stefan’s problems and scattering connection with new structures.

1. Introduction – Mathematics in Industrial Sub branch

Machine building – Casting and Heat treatment

The processes of structures formation are: phase transitions of I\(^{\text{st}}\) order (casting) and II\(^{\text{nd}}\) order (heat treatment); elastic and plastic deformation. The type of these structures is polycrystalline grains with size of macro-scale to 1 \(\mu\)m or micro-scale below 1 \(\mu\)m under a lattice parameter in Å. The structure of all materials is a winner of its properties and the based interest of industries is: 1. improving the working properties of known materials; 2. creating new materials by structures design.

Mathematical description of the phase transitions [1-8, 38 and 40] presented by: the theory of thermal conductivity with the tasks of Stefan and Stefan-Schwarz; the fundamental equation of the formation of new phases of Kashchiev.

On the Fig.1 we introduced castings technologies with different velocity of solidification and science support of micro-foundry

![Diagram](image)

**Fig. 1** Industrial mathematics micro-foundry. Solidification process in temperature zone and character temperatures (TS, TLS and TL, where solid S and liquid L): 1 air craft engine with complex geometry; 2 rapid solidification metallic glass; 3 scattering of latent heat of phase transition in solidification point (zone) – structure design, crystallization, velocity, computational physics, mathematical physics, fundamental and applied mathematics.

Electronic structure is origin of all properties of the metals and its alloys. For description the structures and properties of the metals and alloys are approach mathematics and mathematical physics. It will not be wrong to say that we need a nearly full knowledge i.e. tasks Stefan and Stefan-Schwarz must include mathematics and mathematical physics. This is done through mathematical tasks bridges between mathematical fields and quantum mechanics.

An example of such approach is the use of the results of the theory of scattering. In paper [2] is developed classical Stefan’s problem.

In Institute of Mathematics of Bulgarian academy of science are obtained great results about mathematical theory of scattering from Veselin Petkov [10] and Vladimir Georgiev [11].

The first base problem of scattering theory is proving existence of scattering operator. The second direction of development is inverse problem.

It is well known, that thermodynamics driving force of the nucleation at phase transitions of I\(^{\text{st}}\) order (crystallization: liquid (L) ↔ solid (S)) is \(\Delta \mu = \mu_S - \mu_L = Q_L\ln(T_m/T)\), where \(\mu_S(S)\) are thermodynamics potentials of base (matter) and new phases; \(T_m\) and \(T\) are temperatures of transition and supercooling of the base (L) phase; \(Q_L\) – latent heat of melting. The supercooling is \(\ln(T_m/T) = \Delta T_r + \Delta T_k + \Delta T_s\), where \(\Delta T_r\) – supercooling effect radius of curvature, \(\Delta T_k\) – supercooling for the transfer of atoms in the interface of liquid-solid phase; \(\Delta T_s\) – supercooled liquid phase, where scattering heat of the phase transition. Received development task of Stephen with scatter theory [2] and assessments of supercooled melt \(\Delta T_S = 0, 1, 2, ..., \max K\) in charge of literature [9]. Here must many investigations – measurements and theoretically; but our opinion, that \(\Delta T_S\) is a bridge to Quantum mechanics.

Aim of this paper is to show methodological need to use the full modern knowledge required by the industry examples of casting and heat treatment.

Philosophy is a science with a systematic approach and rational argument, but it does not rely on the scientific method and not strictly defined and accepted subject [17]. Scientific rational argument, but it does not rely on the scientific method.

Philosophy is the study of general and fundamental questions concerning man and the world [17], with mine areas and objects research.

Fundamentals of philosophy: Metaphysics – Nature and Origin of the existing and the world; Ontology – Being; Epistemology – Knowledge of nature and the possibility of cognitive process; Ethics – Morality – how to act human, correct behavior and “good life”; Political philosophy – Governance and respect for human and communities to the state; Aesthetics – beautiful, sublime, art, pleasure; Logic (mathematical and philosophical) – Forms and laws of thinking; valid forms of argumentation; Philosophy of Language: Beginning, development, use and attitudes towards thinking; Scientific methodology (academic disciplines) – Grounds and subject: science; history; mathematics; physics; psychology; anthropology; etc.

Philosophical achievements and results strongly influence [17]: the development of a given society; educational institutions and practices. The results of philosophy are a way of mediation in their application in academic and scientific disciplines [17]: as the importance of logic in mathematics, linguistics, psychology and computer science; generalized the key role of the philosophy of science through scientific methodology. The philosophy has many branches and one of them is the philosophy of science [17] with branches: philosophy of mathematics; philosophy of physics; philosophy of biology, etc. Philosophy of science is divided and developed very intensive branches in the 19th and 20th centuries, which continues to this day. It is well known that mathematics is separated from philosophy in 1600year after Christ.

2.2 Philosophy of Mathematics holds a special place in the philosophy of science

The philosophy of mathematics – subject: studying philosophical assumptions, foundations and implications of mathematics; the need to present the nature and methodology of mathematics to understand the place of mathematics in people’s lives [19, 20, 21, 22].

Reasons for the emergence of mathematics are the need for description of reality first appeared historically known mathematical knowledge is ~ 2500year before Christ (to new era) [26, 27, 28]. In antiquity it was gradually realized the abstract nature of mathematical objects. Mathematics differs from the humanities and natural sciences [19, 20, 21, 22] for example: 1. The objects of study in the natural sciences are located in space and in time, which is not at all clear that applies to objects in mathematics; 2. Methods of study of mathematics very different from the methods of the natural sciences. The use of mathematics in the humanities and natural sciences is due to: there are mathematical properties and regularities of reality, and any object has some mathematical properties. Therefore any theory in natural sciences is mathematized because it contained mathematical knowledge. For example, physics is heavily mathematized.

Explanatory notes on some results on the philosophy of mathematics we present according to works [19, 20, 21, 22]: Mathematical realism: there are mathematical structures, regardless of the human mind; Mathematical anti-realism: mathematical statements are true values, but they are not corresponding to a specific area of intangible or non-empirical objects; Mathematical Platonism: a form of realism, showing that mathematical structures are abstract entities having no spatial and temporal or causal properties, and are eternal and immutable; Platonism of Gödel posits a special kind of mathematical intuition, allowing to perceive mathematical objects directly; Full-blooded Platonism is a modern response to the fact different sets (groups) of mathematical structures can be proven to exist according to the axioms and inference rules employed (for example, the right of excluded middle and the axiom of choice); Empiricism is a form of realism that denies that mathematics can be known a priori at all; Mathematical monism Max Tegmark’s hypothesis mathematical universe if there are any mathematical objects, they exist physically. There all mathematical objects, they exist physically; Logicism is thesis boils down to mathematics, logic, and hence nothing but a part of logic; Formalism argues that mathematical statements may be thought of as statements about the consequences of certain rules of strings; Conventionalism convectional, conditional; Psychologism is the position that mathematical concepts and/or truths are based on materials derived from or explained by psychological facts (or laws); In mathematics, intuitionism is a program of methodological reform whose motto is that “there are no non-experienced mathematical truths” (L.E.J. Brouwer); Constructivism includes regulative principle that only mathematical structures that cannot be explicitly constructed in a way to be admitted to mathematical discourse; Finitism is an extreme constructivism, according to which a mathematical object does not exist, unless it can be made of natural numbers in a finite number of steps; Ultrafinitism is even more extreme version of finitism, which rejected not only infinities, but limited quantities that can actually be constructed with available resources. Structuralism is position, considering that mathematical theories describe structures and that mathematical objects are exhaustively defined by their places in such structures, therefore, has no the intrinsic properties; Embodied mind theories (perfectly realized mind theories) claim that mathematical thinking is a natural product of the cognitive apparatus of the man who is in our physical universe.

Mathematics and mathematical knowledge are used by the methodology of mathematics in a concrete science. This is achieved by applying the philosophy of mathematics in the philosophy of science for specific use of the methodology of mathematics.

Mathematical monism we believe is a heavy time for mathematicians. They require a huge amount of experimental data and described by mathematics. We think that should be used and develop multi-scales approach.

2.3 Foundations of Mathematics [23, 24 and 25]

Hallmark of mathematics is its logical rigor: it deals with precisely defined concepts and proven safe allegations. Mathematical definitions are comprehensive: they contain the necessary and sufficient conditions under which an object can be assigned to the volume of identifiable concept. This attribute mathematical concept is suitable for preparation of mathematical statements whose meaning is quite clear. About the veracity of these claims can be checked so that the resulting conclusions to be absolutely sure. Mathematical assertions, once established, are outside any possible claim. This distinguishes mathematics from both the natural and the humanities, whose allegations can be substantiated most beyond any reasonable doubt, but beyond any doubt at all. This explains the fundamental role of proof in mathematics. Like any science and mathematics is developed, which includes among other things, the existence of unresolved issues. But mathematics is very clear distinction between hypotheses (i.e., statements which may sound plausible, but still not completely certain) and theorems (i.e. statements that are proven and rigorous about their authenticity longer doubt).

In addition to establishing the veracity of claims proof has another function: with its help investigate links between
claims - an indispensable tool in mathematics since its object includes not only mathematical truths, but their connections. In addition to establishing the veracity of claims the proof has another function: with its help investigate links between claims - an indispensable tool in mathematics since its object includes not only mathematical truths, but their connections. However, mathematics is not a science of evidence such science is logic [35].

Causality or causal connection (more causality, causality and modality) is called the relationship between one event (cause) and another event called effect, where the second event is understood as a consequence of the first [29, 30, 31, 32, 33 and 37]. In ordinary use the concept of causality can refer to the connection of several factors (reasons) and a phenomenon. Anything that affects an effect is a factor of this effect. Direct factor called factor that achieves on effect directly i.e. without the involvement of other factors ("intermediate variables "). The factor called factor that achieves on effect directly i.e. without the involvement of other factors ("intermediate variables "). The involvement of other factors ("intermediate variables "). The most accurate science "mathematics" is experiencing methodological difficulties to define itself. The definition of Aristotle that mathematics is the science of the other. Anyone not solved a mathematical problem (task) is not actually "cancel" and is highly illuminated and attack (many time!) to obtain the result. Incompleteness of definitions cannot be filled by a list of mathematical disciplines, as this list is constantly changing, constantly arise in the mathematics new areas.

Definition: Mathematics contains mathematical knowledge, foundations of mathematics, methodology of mathematics and philosophy of mathematics in a complex interconnectivity and continuous development.

3. Mathematical Physics – Applied Mathematics

Principle of causality is a fundamental principle in physics, which states that any event that takes place in a physical system may influence this system in the future, but not to influence her behavior in the past. This means, that if the two events are separate, it neither can be no reason, no consequence to the other.

3.1 Phase transition of first order – thermodynamics, theory of heat conductivity

Mathematical fundamental theories are: Stefan and Stefan-Schwartz problems

- equations- conductivity of liquid L and solid S phases

\[ \frac{\partial T_{L}}{\partial t} = a_{L} \frac{\partial^{2} T_{L}}{\partial x^{2}}, \quad \frac{\partial T_{S}}{\partial t} = a_{S} \frac{\partial^{2} T_{S}}{\partial x^{2}} \quad \text{ (1, 2)} \]

-initial condition of liquid phase (L)

\[ t=0: \quad T_{L}(x, t) = T = \text{const}_L \] (1, 3)

-boundary conditions

\[ x = \infty \quad \text{and} \quad t \geq 0 \quad T_{L}(x, t) = T = \text{const}_L \] (1, 4)

\[ x = 0 \quad \text{and} \quad t \geq 0 \quad T_{S}(x, t) = T = \text{const}_S \] (1, 5)

-boundary conditions at the moving interfacial surface (or solidification (front))

\[ x_{fr} \quad \text{and} \quad t \geq 0 \quad T_{L}(x_{fr}, t) = T_{S}(x_{fr}, t) = \text{const}_m \] (1, 6)

-heat balance of \( x_{fr} \)

\[ X_{fr} : \quad -\lambda_{L} \frac{\partial T_{L}}{\partial x} \bigg|_{x_{fr}} = -\rho_{L} \frac{Q_{fr}}{C_{fr}} \frac{\partial T_{L}}{\partial t} - \lambda_{S} \frac{\partial T_{S}}{\partial x} \bigg|_{x_{fr}} = \rho_{S} C_{fr} \frac{\partial T_{S}}{\partial t} \quad \text{ (1, 7)} \]

The solution is for \( K, T_{L}, T_{S} \) and \( x_{fr} \):

\[ T_{L} = T' - \left( T' - T_{m} \right) e^{-\frac{K}{2\sqrt{a_{L}}}} \]

\[ T_{S} = T'' + \left( T'' - T_{m} \right) e^{-\frac{K}{2\sqrt{a_{S}}}} \]

Stefan-Schwartz problem:

\[ \frac{\partial T_{M}}{\partial t} = a_{L} \frac{\partial^{2} T_{M}}{\partial x^{2}}, \quad \frac{\partial T_{L}}{\partial t} = a_{L} \frac{\partial^{2} T_{L}}{\partial x^{2}}, \quad \frac{\partial T_{S}}{\partial t} = a_{S} \frac{\partial^{2} T_{S}}{\partial x^{2}} \]

\[ \text{- initial conditions liquid and mould} \]

\[ t=0: \quad T_{L}(x, t) = T = \text{const}_L \]

\[ T_{S}(x, t) = T = \text{const}_S \]

\[ 0 \leq t \quad T_{m}(x, t) = T = \text{const}_m \] (2, 4)

-boundary conditions melt, a solid phase and contact (index C) (solid phase/mould)

\[ x = \infty \quad \text{and} \quad t \geq 0 \quad T_{m}^{C}(x, t) = T'' = \text{const}_m \]

\[ x_{fr} \quad \text{and} \quad t \geq 0 \quad T_{S}^{C}(x, t) = T'' = T_{C}(0, t) = T'' \]

\[ \text{- mould} \]

\[ x = \infty \quad \text{and} \quad t \geq 0 \quad T_{M}^{C}(x, t) = T'' = \text{const}_m \] (2, 8)

-boundary conditions at moving (solidification (front)) \( x_{fr} \) and heat balance is

\[ X_{fr} : \quad -\lambda_{L} \frac{\partial T_{L}}{\partial x} \bigg|_{x_{fr}} = -\rho_{L} \frac{Q_{fr}}{C_{fr}} \frac{\partial T_{L}}{\partial t} - \lambda_{S} \frac{\partial T_{S}}{\partial x} \bigg|_{x_{fr}} = \rho_{S} C_{fr} \frac{\partial T_{S}}{\partial t} \quad \text{ (2, 9)} \]

The solution is the equations for \( T_{C}, K, T_{L}, T_{S}, T_{M} \) and \( x_{fr} \):

\[ T_{C} = \frac{T_{m}}{1 + \frac{b_{M}}{b_{S}} e^{-\frac{K}{2\sqrt{a_{S}}}}} \]

\[ 59 \quad \text{YEAR V, ISSUE 2, P.P. 57-63 (2017)} \]
\[
\rho_L Q_m \frac{\sqrt{\pi}}{2} K = b_L \left( T_m - T_C \right) e^{\frac{K}{2a_L}} - b_L \left( T' - T_m \right) e^{\frac{K}{2a_L}}, \quad (2, 11)
\]

\[
T_L = T_m - \left( T' - T_m \right) \frac{1 - \text{erf} \left( \frac{x}{2\sqrt{a_L}t} \right)}{1 - \text{erf} \left( \frac{K}{2\sqrt{a_L}} \right)}, \quad (2, 12)
\]

\[
T_S = T_C + \left( T_m - T_C \right) \frac{\text{erf} \left( \frac{x}{2\sqrt{a_S}t} \right)}{\text{erf} \left( \frac{K}{2\sqrt{a_S}} \right)}, \quad (2, 13)
\]

\[
T_M = T_C - \left( T_m - T_C \right) \text{erf} \left( \frac{x}{2\sqrt{a_M}t} \right), \quad (2, 14)
\]

\[
X_f = K \sqrt{t}. \quad (2, 15)
\]

In tasks (1) and (2) use the following symbols, \( a_L, a_M, a_s \) and \( a = \sqrt{\lambda c / \rho} \) are thermo conductivity numbers and \( \lambda, c, \rho \) are thermal conductivity, heat capacity and density of liquid (L), solid (S), mould (M) materials; \( b = \sqrt{\lambda c \rho} \) and \( b_L, b_S, b_M \) are capability accumulation material (L), (S) and (M); \( K \) is coefficient of solidification; \( T_{LSM} \) is temperature of cast phases (L), (S) and (M); \( T_{Ct} \) is contact temperature between cast and mould, and the same, but \( T_{S_Ct}, T_{M_Ct} \) at \( x=0 \); \( t \) is time; \( Q_m \) - latent heat of melting; \( x \) is coordinate.

The tasks of Stefan’s type are verified experimentally and used in the creation of mathematical models for technology phase transformations. At high cooling rates and small size of the studied open system these tasks provide huge opportunity for creating mathematical models. This methodological approach is: estimates of temperature fields and thermal balance.

This methodology based on the task of Stefan we can applied of process and the phase transition of the second order (heat treatment). It should consider the specific difference between the two phase transitions. The description of the temperature field in the phase transition of the first kind is more complicated in comparison with the temperature field of phase transition of the second order.

For example we have chosen the application of task Stephen-Schwartz for thermodynamics open system from aluminum melt and steel mould – phase transition of first order. The thickness of cast and mould each is with a size of 100 microns. To calculate the temperature field of the rapid solidification process of molten pure aluminum in quenching steel mould is on the Fig. 2 [38] – comparative analysis between the analytical solution and numerical solution with the Finite element method of Stephen-Schwartz problem.
For the process of heat transfer from Figure 2 we compare the ideal and the real contact via the contact temperatures $T_{IC}^{C}$ and $T_{Real}^{C} = (T^M_2 + T^S_2)/2$ and values of $x_p$ from the analytical $X_{flux}^{Analytical}$ and the numerical $X_{flux}^{Numerical}$ solutions.

Contact temperatures: $T_{IC}^{C} = 229.15 \, ^{o}C$ and $T_{Real}^{C} = 270.9 \, ^{o}C$ are different with 18.22 %.

Heat balance: $X_{flux}^{Analytical} = 32.55 \, \mu m$ and $X_{flux}^{Numerical} = 32.653 \, \mu m$ have differences 0.316 %.

Pure metals and part of the alloys solidified with one interface surface and other alloys solidified in temperature interval between two moving interface surfaces. Foundry engineering methodology: 1) Experimental measurement and numerical description of non-stationary temperature field with moving liquid/solid (L/S) interfacial surface(s) and constant temperature; 2) Microscopic measurements of the obtaining structure and a description of the crystallization (structure formation) processes.

Summary amending heat at the expense of conductivity
Summary amending the heat at the expense of moving the boundary
Summary amending the heat on account of losses to the environment

The solidification (macro level of phase transition first order) has two typical cases on Fig. 3: 1. Equilibrium – without supercooled of metal melt; 2. Non-equilibrium – Low or High supercooled of metal melt. In both cases of solidification is released the latent heat of transition (melting) $Q_m$.

$$t_f = \frac{\Delta T_s}{GR}, \quad \Delta U = \frac{c_s \Delta T}{Q_m}. \quad (3, 4)$$

It should be noted in equations (3, 4) we only have thermodynamic variables and time. The supercooling is very important, but it many difficult is defined. Factors that affect supercooling are many and relate to the mechanism of structure formation of each material (metal, alloy, etc.). The supercooling of eq.34 as local conditions is similar to similar to supercooling $\ln(T_m/T)$ (see the introduction).

Latent heat of phase transition is scattering in volume with supercooled melt [9] i.e. connectivity between the release of latent heat of melting and its scattering in supercooled melt is applied the theory of diffusion of works [10, 11]. In [2] is a developed Stefan task considering the supercooled area, which disperses and releases latent heat $Q_{m}$ On Fig. 4 is shown only temperature field [4] of supercooled zone of pure Al.
Mathematical Education – Complete orientation (talented education in mathematics in the direction of maximum (almost full) math education and retraining that obliges the development of technology based on research results. Part of the Industry 4 and Mathematics has become the language of science and others) and released millions of jobs,

created by the famous Bulgarian mathematician Blagovest Sendov new educational programs. An example is the educational system, in many educational systems scientists mathematicians develops orientation of the student (child or adult). In this direction, working which are often called subversive effect. For these reasons, society The change is exponential rate, close to many branches everything [12, 13 and 14].

Industry 4 requires a complete mathematization of commodity and opens a permanent market for it.

appears acute need for training and retraining of large groups of people, and of course we have education for all life.

Mathematics needed to describe the process of phase transition and the formation of structures corresponds to the basics (pure) mathematics.

Modern technologies require mathematical physics.

4. Industry mathematics – Industry 4

The fundamental mathematics, mathematical physics and fundamental experimental physics lead to a revolution in physics [34]. Based on the results of fundamental research are create new technologies.

Industry 4 is a comprehensive revolutionary change of society by the advent of technology [16]: artificial intelligence, robotics, internet things, autonomous vehicle without a driver, 3D printing, nanotechnology, biotechnology, materials science, energy storage and quantum computing. These new technologies are the foundation for creating "smart factories". New technologies are related to the requirements of ecology; circulation of materials on the market - production - use - scrap for production; renewable energy and renewable energetics. Knowledge becomes a commodity and opens a permanent market for it.

Industry 4 requires a complete mathematization of everything [12, 13 and 14].

The change is exponential rate, close to many branches and released millions of jobs, which are often called subversive effect. For these reasons, society appears conditions for chaos, a huge challenge for man.

Appears acute need for training and retraining of large groups of people, and of course we have education for all life.

5. Mathematical Education – Complete orientation (talented and others)

Mathematics has become the language of science and technology based on research results. Part of the Industry 4 and math education and retraining that obliges the development of education in mathematics in the direction of maximum (almost full) orientation of the student (child or adult). In this direction, working in many educational systems scientists’ mathematicians develops new educational programs. An example is the educational system, created by the famous Bulgarian mathematician Blagovest Sendov called “Sendov’s system” [36].

Program for lifelong learning is developed in Europe. Universität Wien, Dipartamento di Matematica, Universität di Pisa, VIA University College – Læreruddannelsen i Århus and Institute of Mathematics and Informatics at the Bulgarian Academy of Sciences was held seminar [39]. The importance of this workshop is wider than the specific reports. This is to answer the question "how methodologically to help in the economy by math (single person or micro-business)?”.

6. Conclusions

The assessment of overcooling is a quantitative relationship and driving force of crystallization processes and structures.

Mathematics needed to describe the process of phase transition and the formation of structures corresponds to the basics (pure) mathematics.

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