LIMITATIONS TO SUSPENSION PERFORMANCE IN A TWO-DEGREE-OF-FREEDOM CAR ACTIVE SUSPENSION

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Abstract: *It is often assumed that if practical difficulties are neglected, active systems could produce in principle arbitrary ideal behavior. This paper presents the factorization approach that is taken to derive limitations of achievable frequency responses for active vehicle suspension systems in terms of invariant frequency points and restricted rate of decay at high frequencies. The factorization approach enables us to determine complete sets of such constraints on various transfer functions from the load and road disturbance inputs for typical choices of measured outputs and then choose the "most advantageous" vector of the measurements from the point of view of the widest class of the achievable frequency responses. Using a simple linear two degree-of-freedom car suspension system model it will be shown that even using complete state feedback and in the case of in which the system is controllable in the control theory sense, there still are limitations to suspension performance in the fully active state.*

Keywords: VEHICLE, ACTIVE SUSPENSION, CONTROL, LIMITATIONS, PARAMETRIZATION

1. Introduction

Two major performance requirements of suspension are to improve ride and handling quality when random road and load disturbances from the environment act upon running vehicles. Automotive suspensions are designed to provide good vibration insulation of the passengers and to maintain adequate adherence of the wheel for braking, accelerating and handling, i.e. the purpose of active suspensions in terms of performance is to improve both of these conflicting requirements.

In this paper, it will be shown the factorization approach taken to derive limitations of achievable frequency responses for active vehicle suspension systems. As we will see, limitations derived for a traditional one-quarter-car model (Fig.1) in the frequency domain arise in the form of invariant frequency points and restricted rate of decay at zero and infinite frequencies.

Youla-Kucera factorization approach to feedback system stability has been shown in [2], [3] to derive achievable dynamic responses for active suspension systems of vehicles. Complete sets of constraints on various transfer functions from the road and load disturbance inputs were derived for typical choices of measured outputs.

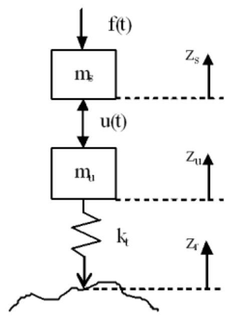

Fig. 1. *One-quarter-car model*

The approach was illustrated for the one-quarter-car model shown in Fig.1, where:

- *u(t)* control input (active suspension force) [N]
- m_u weight of the unsprung mass (wheel) [kg]
- m_s weight of the sprung mass supported by each wheel and taken as equal to a quarter of the total body mass [kg]
- k_t stiffness of the tire [N/m]
- $z_r(t)$ road displacement (road disturbance) [m]
- $z_s(t)$ displacement of the sprung mass [m]
- $z_u(t)$ displacement of the unsprung mass [m]
- *f(t)* load disturbance [N]

Note that if the one-quarter-car model contains also a passive suspension system (a sprung of stiffness k and a shock absorber of damping quotient b then the active suspension force u(t) involves also the adequate force generated by the passive suspension system.

On the base of the Youla-Kucera parametrization, complete sets of limitations were derived for transfer functions from the road disturbance input (load f(t) is absent):

$$H_{zw}^1(s) = \frac{Z_s(s)}{Z_r(s)} / _{f=0} \quad \text{(to the sprung mass position)} \quad (1)$$

$$H_{zw}^2(s) = \frac{Z_s(s) - Z_u(s)}{Z_r(s)} / _{f=0} \quad \text{(to the suspension deflection)} \quad (2)$$

$$H_{zw}^3(s) = \frac{Z_u(s) - Z_r(s)}{Z_r(s)} / _{f=0} \quad \text{(to the tire deflection)} \quad (3)$$

and analogically for the load disturbance input for various choices of measured outputs even for full state feedback.

2. Comprime Factorization

Consider the standard feedback configuration shown in Fig.2,

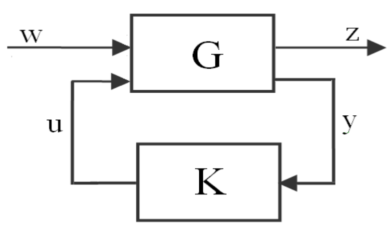

Fig. 2 *Standard feedback configuration*

where *w* is the exogenous input, typically consisting of disturbances and sensor noises, *u* is the control signal, *z* is the output to be controlled, and *y* the measured output. In general, *u, w, y,* and *z* are vector-valued signals.

The transfer matrices *G(s)* and *K(s)* are, by assumption, real-rational and proper: *G* represents a generalized plant, the fixed part of the system, and *K* the controller [4]. Partition *G(s)* as:

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \quad (4)$$

Then Fig.2 stands for the following algebraic equations:

$$Z(s) = G_{11}(s)W(s) + G_{12}(s)U(s) \tag{5}$$

$$Y(s) = G_{21}(s)W(s) + G_{22}(s)U(s) \tag{6}$$

$$U(s) = K(s)Y(s) \tag{7}$$

Manipulating the equations listed above, the following transfer function $T_{zw}(s)$ from w to z as a linear-fractional transformation of $K(s)$ can be derived:

$$T_{zw} = G_{11} + G_{12}K[I - G_{22}K]^{-1}G_{21} = \\ = G_{11} + G_{12}[I - KG_{22}]^{-1}KG_{21} \tag{8}$$

It is shown in [1] that the set of all proper real-rational matrices $K(s)$ stabilizing $G(s)$ is parametrized by a free parameter $Q(s) \in RH_\infty$ as follows:

$$K = [Y - MQ][X - NQ]^{-1} = \\ = \left[\tilde{X} - Q\tilde{N}\right]^{-1}\left[\tilde{Y} - Q\tilde{M}\right] \tag{9}$$

where:

$M(s), N(s), X(s), Y(s) \in RH_\infty$, and $\tilde{M}(s), \tilde{N}(s), \tilde{X}(s), \tilde{Y}(s) \in RH_\infty$ can be found by comprime factorization approach of $G_{22}(s)$:

$$G_{22}(s) = N(s)M^{-1}(s) = \tilde{M}^{-1}(s)\tilde{N}(s) \tag{10}$$

$$\begin{bmatrix} \tilde{X}(s) & -\tilde{Y}(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix}\begin{bmatrix} M(s) & Y(s) \\ N(s) & X(s) \end{bmatrix} = I \tag{11}$$

Substituting the equation (9) into (8) we obtain the transfer matrix $T_{zw}(s)$ from w to z in terms of the free parameter $Q(s) \in RH_\infty$:

$$T_{zw}(s) = \\ = G_{11}(s) + G_{12}(s)M(s)\left[\tilde{Y}(s) - Q(s)\tilde{M}(s)\right]G_{21}(s) = \\ = G_{11}(s) + G_{12}(s)[Y(s) - M(s)Q(s)]\tilde{M}(s)G_{21}(s) \tag{12}$$

As the parameter $Q(s)$ varies over the set of all stable proper functions, the equation (12) parametrizes all achievable transfer functions $T_{zw}(s)$.

If it is assumed that the tire does not leave the ground, for the one-quarter car model (Fig.1) the linear differential equations of motion are:

$$m_s\ddot{z}_s = u - f \tag{13}$$

$$m_u\ddot{z}_u = -u + k_t(z_r - z_u) \tag{14}$$

where z_u and z_s are measured from the static equilibrium position.

First, let the load disturbance is absent ($f=0$). Adding equations (13) and (14) we obtain the invariant equation of:

$$m_s\ddot{z}_s + m_u\ddot{z}_u = k_t(z_r - z_u) \tag{15}$$

that is independent on the suspension force u. The following transfer functions will be investigated:

$$H_{SP}(s) = Z_s(s)/Z_r(s) \tag{16}$$

$$H_{SD}(s) = [Z_s(s) - Z_u(s)]/Z_r(s) \tag{17}$$

$$H_{TD}(s) = [Z_u(s) - Z_r(s)]/Z_r(s) \tag{18}$$

3. Invariant Properties

Manipulating the equation (15) we can derive the following invariant identities:

$$[(m_u + m_s)s^2 + k_t]H_{SP}(s) - [m_u s^2 + k_t]H_{SD}(s) = k_t \tag{19}$$

$$m_s s^2 H_{SP}(s) + [m_u s^2 + k_t]H_{TD}(s) = -m_u s^2 \tag{20}$$

$$[(m_u + m_s)s^2 + k_t]H_{TD}(s) + m_s s^2 H_{SD}(s) = \\ = -(m_u + m_s)s^2 \tag{21}$$

It is obvious from (19) and (20) that the sprung mass position transfer function H_{SP} has an invariant "tire-hop" frequency at $\omega_1 = \sqrt{k_t/m_u}$, where:

$$H_{SP}(s)/_{s=j\omega_1} = -m_u/m_s \tag{22}$$

Similarly, from (16) and (18) the suspension deflection transfer function H_{SD} has an invariant "rattle-space" frequency at $\omega_2 = \sqrt{k_t/(m_u + m_s)}$ and:

$$H_{SD}(s)/_{s=j\omega_2} = -(1 + m_u/m_s) \tag{23}$$

Finally, from (20) and (21), the tire deflection transfer function H_{TD} does not have any invariant frequency point except $\omega_3 = 0$, where:

$$H_{TD}(s)/_{s=j\omega_3} = 0 \tag{24}$$

4. Transfer Functions Limitations

In the next, consider the standard block diagram shown in Fig.2.

As an example, consider:

$$w=z_r\ ,\ z=z_s \qquad \underline{y} = [\dot{z}_u, z_s - z_u, z_u - z_r]^T.$$

Then:

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \\ = \begin{bmatrix} 0 & \dfrac{1}{m_s s^2} \\ \begin{bmatrix} \dfrac{sk_t}{m_u s^2+k_t} \\ \dfrac{-k_t}{m_u s^2+k_t} \\ \dfrac{-m_u s^2}{m_u s^2+k_t} \end{bmatrix} & \begin{bmatrix} \dfrac{-s}{m_u s^2+k_t} \\ \dfrac{(m_u+m_s)s^2+k_t}{m_s s^2(m_u s^2+k_t)} \\ \dfrac{-1}{m_u s^2+k_t} \end{bmatrix} \end{bmatrix} \tag{25}$$

The limitations of all achievable closed-loop transfer functions $T_{zw}(s) = H_{SP}(s)$ are derived from the right and left comprime factorization of $G_{22}(s)$, i.e.:

$$G_{22} = \underbrace{\begin{bmatrix} \dfrac{-m_s s^3}{p_4(s)} \\ \dfrac{(m_u + m_s)s^2 + k_t}{p_4(s)} \\ \dfrac{-m_s s^2}{p_4(s)} \end{bmatrix}}_{N(s)} \underbrace{\left[\dfrac{m_s s^2(m_u s^2 + k_t)}{p_4(s)}\right]^{-1}}_{M^{-1}(s)} = \\ = \underbrace{\begin{bmatrix} \dfrac{m_s s}{p_2(s)} & \dfrac{m_s s^2}{p_2(s)} & 0 \\ 0 & \dfrac{m_u m_s s^2}{p_2(s)} & \dfrac{-m_s k_t}{p_2(s)} \\ \dfrac{-1}{p_2(s)} & 0 & \dfrac{s}{p_2(s)} \end{bmatrix}^{-1}}_{\tilde{M}^{-1}} \underbrace{\begin{bmatrix} \dfrac{1}{p_2(s)} \\ \dfrac{m_u+m_s}{p_2(s)} \\ 0 \end{bmatrix}}_{\tilde{N}(s)} \tag{26}$$

where $p_2(s)$ and $p_4(s)$ are Hurwitz polynomials of degree 2 and 4, respectively. Then:

$$H_{SP}(s) = \\ = G_{11}(s) + G_{12}(s)M(s)\left[\tilde{Y}(s) - Q(s)\tilde{M}(s)\right]G_{21}(s) = \\ = -\frac{1}{p_4(s)}\tilde{Y}(s)\begin{bmatrix} -sk_t \\ k_t \\ m_u s^2 \end{bmatrix} + \frac{s(m_u s^2+k_t)}{p_4(s)}Q^*(s) \tag{27}$$

where $Q^*(s) = \left[\frac{Q_3(s)}{p_2(s)}\right]$, $Q(s) = [Q_1(s)\ Q_2(s)\ Q_3(s)]$.

It follows from (27) that thanks to the term $\left[\tilde{Y}(s) - Q(s)\tilde{M}(s)\right]$ bounded for large s:

$$\lim_{s\to\infty} s^2 H_{SP}(s) < \infty \tag{28}$$

i.e., the resulting rate of decay is of second degree:

$$H_{SP}(s)/_{s\to\infty} = O(s^{-2}) \quad (29)$$

It is obvious from (27) that the member $\frac{s(m_u s^2+k_t)}{p_4(s)}$ has two imaginary axis zeros - at s=0 and $s = j\omega_1 = j\sqrt{k_t/m_u}$- which can not be canceled by the denominator of $Q^*(s) \in RH_\infty$. With respect to (27) and the Bezout identity (11) it follows, that:

$$H_{SP}(s)/_{s=0} = -\frac{1}{p_4(s)}\tilde{Y}(s)\begin{bmatrix}-sk_t\\ k_t\\ m_u s^2\end{bmatrix}/_{s=0} =$$

$$= -\frac{k_t}{p_4(s)}\tilde{Y}_2(s)/_{s=0} = 1 \quad (30)$$

and similarly:

$$H_{SP}(s)/_{s=j\omega_1} = -\frac{1}{p_4(s)}\tilde{Y}(s)\begin{bmatrix}-sk_t\\ k_t\\ m_u s^2\end{bmatrix}/_{s=j\omega_1} =$$

$$= -m_u/m_s$$
(31)

This result endorses (25).

Thanks to the first order of the imaginary axis zero at s=0, the first derivative of the transfer function *$H_{SP}(s)$* does not have any similar restrictions at this point. Expressions (29), (30) and (31) create the complete set of limitations which any admissible transfer function $H_{SP}(s) \in RH_\infty$ must satisfy. Another words, if any complex transfer function satisfies the mentioned limitations, there exists a stabilizing controller *K(s)* so that $T_{zw}(s) = H_{SP}(s)$. It does not depend on what variables are chosen as the measured output - the limitations always arise in the form of invariant frequency points as was shown in paragraph 3 and in the form of restricted rate of decay at infinite frequencies as shown in (29).

Complete sets of limitations for the transfer functions *$H_{SD}(s)$* and *$H_{TD}(s)$* can be similarly carried out from the corresponding transfer functions *G(s)* or using (26), (27), (28) and the corresponding invariant equation stated above. That way the following complete sets of limitations can be derived:

$$H_{SD}(s)/_{s\to\infty} = O(s^{-2}) \quad (32)$$

$$H_{SD}(s)/_{s=0} = 0, \qquad H_{SD}(s)/_{s\to 0} = O(s) \quad (33)$$

$$H_{SD}(s)-_{s=j\omega_2} = -\left(1+\frac{m_u}{m_s}\right), \text{ where}$$

$$\omega_2 = \sqrt{k_t/(m_u+m_s)} \quad (34)$$

$$H_{TD}(s)/_{s\to\infty} = -1 + O(s^{-2}) \quad (35)$$

$$H_{TD}(s)/_{s=0} = 0,$$
$$H_{TD}(s)/_{s\to 0} = -(m_u+m_s)s^2/k_t + O(s^3)$$
(36)

Note, that even though it is desirable to prevent amplitudes of the frequency responses $H_{SP}(s)$, $H_{SD}(s)$, and $H_{TD}(s)$ being too large in any frequency domain, a brief analysis of the expressions (29) - (36) enables to find out that the investigated transfer functions must have modulus strictly grater that one at some frequencies what indicates the fact that the road disturbance signal is amplified at these mentioned frequencies.

The same approach can be used to derive limitations for other transfer functions and various choices of the measured outputs.

It has been shown that the limitations always arise in the form of invariant frequency points (for example $\omega_1 = \sqrt{\frac{k_t}{m_u}}$ for $H^1_{zw}(j\omega_1)$, $\omega_2 = \sqrt{\frac{k_t}{m_u+m_s}}$ for $H^2_{zw}(j\omega_2)$ and $\omega_3 = 0$ for $H^3_{zw}(j\omega_3)$) and in the form of restricted rate of decay at frequencies tending to zero and infinity.

As an example, the complete sets of constraints for transfer functions $H^1_{zw}(s)$, $H^2_{zw}(s)$ and $H^3_{zw}(s)$, when suspension deflection and suspension deflection velocity are measured, is as follows [2],[3]:

$H^1_{zw}(s)/_{s\to\infty} = O(s^{-3})$ (infinite frequency constraint)

$H^1_{zw}(s)/_{s\to 0} = 1 + O(s^2)$ (zero frequency constraint)

$H^1_{zw}(j\omega_1) = -\frac{m_u}{m_s}$ for $\omega_1 = \sqrt{\frac{k_t}{m_u}}$

and analogically:

$H^2_{zw}(s)/_{s\to\infty} = -\frac{k_t}{m_u}s^{-2} + O(s^{-3})$ (infinity freq. constraint)

$H^2_{zw}(s)/_{s\to 0} = O(s^2)$ (zero frequency constraint)

$H^2_{zw}(j\omega_2) = -(1+\frac{m_u}{m_s})$ for $\omega_2 = \sqrt{\frac{k_t}{m_u+m_s}}$ and

$H^3_{zw}(s)-_{s\to\infty} = -1 + \frac{k_t}{m_u}s^{-2} + O(s^{-3})$(infinity freq. constraint)

$H^3_{zw}(s)/_{s\to 0} = -\frac{(m_u+m_s)}{k_t}s^2 + O(s^4)$ (zero freq. constraint)

$H^3_{zw}(j\omega_3) = 0$ for $\omega_3 = 0$.

It is often assumed that if practical difficulties are neglected, active systems could in principle produce arbitrary ideal behavior. This paper presents the factorization approach that is taken to derive limitations of achievable frequency responses for active vehicle suspension systems in terms of invariant frequency points and restricted rate of decay at high frequencies.

In control law design for active suspension system of vehicles it is demanded to prevent magnitudes of the road and load frequency responses from being too large. There are some frequency points and frequency ranges where the transfer functions have modulus strictly greater than one i.e. where road and load disturbance amplification occur. On the base of the Bode integral it can be shown that the transfer functions must be greater in modulus to at least the same extend that it is less than one, when measured in terms of the area on a Bode magnitude plot [1], [2] [3]. In such a case there is a possibility to shift frequency ranges where disturbance amplification occurs to a ”more advantageous place” or to make magnitudes lower spreading the frequency range.

5. Analysis of the Complete Sets of Limitations

In context with transfer functions $H(s)$ of the one-quarter-car model given in Section 1, the generalized Bode integral theorem can be modified as follows [1] :

Theorem 1. Let RH_∞ is a set of rational transfer functions that are stable (their poles lie in the open right half-plane) and proper (the numerator degree of these functions is less than or equal to the denominator degree). Let $H(s)$ belongs to RH_∞ and satisfies $H(s)/_{s\to\infty} = -1 + O(s^{-2})$. Let $\{z_i, i = 1,..,n \quad n \in N\}$ are zeros of $H(s)$ with $Re(z_i) > 0$. Then:

$$\int_0^\infty log|H(j\omega)|\,d\omega = \pi\sum_{i=1}^n Re(z_i) \quad (37)$$

In control law design, it is desirable to prevent amplitudes of the dynamic responses $H^1_{zw}(s)$, $H^2_{zw}(s)$ and $H^3_{zw}(s)$ from being too large.

A brief examination of the results stated in Section 1 shows that the suspension deflection transfer function $H^2_{zw}(s)$ has modulus strictly grater than one for $\omega_2 = \sqrt{\frac{k_t}{m_u+m_s}}$ ($H^2_{zw}(j\omega_2) = -\left(1+\frac{m_u}{m_s}\right)$), i.e. at this frequency an amplification grater than one occurs. This amplification can be made less only and only by adjusting the ratio of the unsprung and sprung masses. From the result:

$$H_{zw}^{3}(s)/_{s\to\infty} = -1 + \frac{k_t}{m_u}s^{-2} + O(s^{-3}) \quad (38)$$

it is evident, that $|H_{zw}^{3}(j\omega)|$ must tend to one from above as ω tends to ∞ and it turns out that $|H_{zw}^{3}(j\omega)|$cannot be made less than or equal to one at all frequencies. Since the right-hand side of (37) is non-negative then it is not possible for $|H_{zw}^{3}(j\omega)|$ to be less than or equal to one at all frequencies since that would make the left-hand side of the equation (37) negative. It has been shown in [3], that no matter what signals are used for feedback, the tire deflection transfer function must amplify road disturbances at some frequencies. This fact is valid even for full state feedback used in the control loop.

A similar theorem is valid for transfer functions where $H(s)/_{s\to 0} = 1 + O(s^2)$

Theorem 2. Let $H(s)$ belongs to RH_∞ and satisfies$H(s)/_{s\to 0} = 1 + O(s^2)$.Let $\{z_i, i = 1, .., n \quad n \in N\}$ are zeros of $H(s)$ with $Re(z_i) > 0$. Then:

$$\int_0^\infty log|H(j\omega)|\frac{d\omega}{\omega^2} = \pi\sum_{i=1}^{n}\frac{Re(z_i)}{|z_i|^2} \quad (39)$$

Similarly to the consequences of Theorem 3, the result $H_{zw}^{1}(s)/_{s\to 0} = 1 + O(s^2)$ from Section 1 is the case when $|H_{zw}^{1}(j\omega)|$ cannot be less than or equal to one at all frequencies since that would make the left-hand side of (39) negative. This fact is again valid even when full state feedback is introduced in the control loop [3].

In such cases that were mentioned above designers have the only possibility ”to shift” the frequencies where the amplifications occur to ”more advantageous places” or ”to spread” the ranges where amplifications occur making the amplification lower the positive area of the Bode magnitude plot, i.e. the area where $|H(j\omega)|$ is greater than one (is grater to at least the same extent than the negative area where $|H(j\omega)|$ is less than one) by choosing a proper feedback controller. Analogically, similar results can be derived for arbitrarily chosen measurements and load disturbances.

6. Results

Using a simple linear two-degree of freedom car suspension model in Fig.1, it was shown that there are still some limitations to suspension performance even in the fully active state. It has been shown in the paper that there are some frequency points and frequency ranges where the transfer functions have modulus strictly greater than one i.e. where road and load disturbance amplification occurs. On the base of the Bode integral theorems, it has been shown that the transfer functions must be greater in modulus to at least the same extent that it is less than one when measured in terms of the area on a Bode magnitude plot. In such a case there is a possibility to shift frequency ranges where disturbance amplification occurs to a ”more advantageous place” or to make magnitudes lower spreading the frequency range.

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References

[1] Freudenberg, J. S., Looze, D. P., Right-half Plane Poles and Zeros and Design Tradeoffs in Feedback Systems, IEEE Transactions on Automatic Control, 30/6, 1985

[2] Hyniova, K., Frequency Response Limitations for Active Suspension of Vehicles, Micro-CAD 99 – International Computer Science Conference, Miskolc, pp. 209-217, 2006,

[3] Hyniova, K., Achievable Dynamic Responses for Active Suspension System of Vehicles, Habilitation thesis, Prague, 2000

[4] Zhou, K., Doyle, J.C., Essentials of Robust Control, Prentice Hall, London, 1998