# ANALYTIC SOLUTION OF A NONSTATIONARY EQUATION OF KOLMOGOROV-FELLER TYPE WITH A NONLINEAR DRIFT COEFFICIENT

Prof. Dr. Tech. Sci. Andrei N. Firsov
Peter the Great Saint-Petersburg Polytechnic University, Russia
anfirs@yandex.ru

**Abstract:** *The paper is devoted to the construction of analytical algorithms for solving stochastic integro-differential equations of Kolmogorov-Feller type. Such equations are encountered in problems of control theory, communication theory, stellar dynamics, and so on.*
**Keywords**: KOLMOGOROV-FELLER EQUATION, NONLINEAR DRIFT COEFFICIENT, ANALYTICAL SOLUTION ALGORITHMS

## 1. Introduction

This note is devoted to the construction of an analytical algorithm for solving stochastic integro-differential equations of Kolmogorov-Feller type that occur in problems of control theory, communication theory, stellar dynamics, and so on. In the papers [1 - 5], known to the author and containing variants of analytical and / or numerical algorithms for solving such equations, cases of a linear dependence of the drift coefficient on the spatial coordinate are considered, as a rule. The algorithm of the analytical solution proposed in this paper does not imply such a limitation, and relies on the theory developed earlier by the author of rapidly decreasing generalized functions [6 - 9], where, in particular, the construction of the reconstruction of a "sufficiently fast decreasing function at infinity" is proposed in terms of its power moments. For simplicity of exposition, we restrict ourselves here to the case of a quadratic dependence of the drift coefficient on the spatial coordinate.

## 2. Formulation of the problem

We seek a solution of the equation

$$\frac{\partial W(x,t)}{\partial t}=\frac{\partial}{\partial x}\left[(\alpha x+\beta x^2)W(x,t)\right]+ +\nu\int_{-\infty}^{+\infty}p(A)W(x-A,t)dA-\nu W(x,t) \tag{2.1}$$

under conditions

$$\begin{cases} W(x,0)=W_0(x)-\text{is known function,}\\ W(x,t)\underset{x\to\pm\infty}{\longrightarrow}0 \ \text{ for all } t,\\ \int_{-\infty}^{+\infty}W(x,t)\,dx=1 \end{cases} \tag{2.2}$$

with obvious requirements:

$$p(A)\underset{A\to\pm\infty}{\longrightarrow}0 \ \text{ and } \int_{-\infty}^{+\infty}p(A)dA=1 .$$

We further assume that

$$p(A)=O(\exp(-\gamma|A|^{1+\delta})),\quad |A|\to\infty \tag{2.3}$$

for some $\gamma>0,\quad \delta>0$.

Solution of the problem (2.1) – (2.2) will be sought in the class of "sufficiently fast" decreasing functions (for details, see below).

## 3. About the representation of rapidly decreasing functions through its moments

Let $f(x)$ is continuous, and for some $\lambda>0, \mu>0$

$$f(x)=O(\exp(-\lambda|x|^{1+\mu})),\ |x|\to\infty .$$

Let $C^{(n)}\triangleq\frac{(-1)^n}{n!}\int_{-\infty}^{+\infty}f(x)x^n dx,\ n=0,1,2,\ldots$. Using the methods developed in [7-9], it is comparatively easy to prove the relation

$$f(x)=\frac{1}{\pi}\cdot\frac{1}{\varepsilon}\sum_{n=0}^{\infty}\frac{C^{(n)}}{\varepsilon}\sigma^{\,n}\left(\frac{x}{\varepsilon}\right)+O(\varepsilon),\quad \varepsilon\to 0 \quad .$$

In particular,

$$f(x)=\frac{1}{\pi}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\sum_{n=0}^{\infty}\frac{C^{(n)}}{\varepsilon}\sigma^{(n)}\left(\frac{x}{\varepsilon}\right).$$

Here

$$\sigma^{(k)}\ x\ =\sum_{s=0}^{k}(-1)^s\frac{k!}{(k-s)!}\frac{\sin\left(x+(k-s)\frac{\pi}{2}\right)}{x^{s+1}} .$$

It is not difficult to see that the following equalities hold:

$$\sigma^{(2k)}(x)=(-1)^k\sum_{s=0}^{\infty}\frac{(-1)^s x^{2s}}{(2k+2s+1)(2s)!},$$

$$\sigma^{(2k+1)}(x)=(-1)^k\sum_{s=0}^{\infty}\frac{(-1)^{s+1}x^{2s+1}}{(2k+2s+3)(2s+1)!},$$

$$\sigma^{(2k)}(x)=\frac{(-1)^k}{2k+1}+O(x^2),\quad x\to 0,$$

$$\sigma^{(2k+1)}(x)=O(x),\quad x\to 0,$$

$$\sigma^{(k)}(x)=O\left(\frac{1}{x}\right),\quad |x|\to\infty,$$

$$\sigma^{(k)}\left(\frac{x}{\varepsilon}\right)=\frac{(i)^k}{2}\varepsilon^{k+1}\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}}\tau^k e^{ix\tau}d\tau.$$

## 4. Construction of the solution of the problem (2.1), (2.2)

We put further:

$$C^{(n)}(t)=\frac{(-1)^n}{n!}\left[\int_{-\infty}^{+\infty}x^n W(x,t)\,dx\right].$$

Then from (2.1) we obtain:

$$\frac{d^k}{dt^k}C^{(n)}(t)=\frac{d^{(k-1)}}{dt^{(k-1)}}\left(-\nu C^{(n)}+\nu\sum_{s=0}^{n}p^{(s)}C^{(n-s)}(t)-\alpha nC^{(n)}\right)+ +n(n+1)\beta\frac{d^{(k-1)}}{dt^{(k-1)}}C^{(n+1)}(t),\ n=1,2,3,\ldots;\ k=1,2,3,\ldots . \tag{4.1}$$

$$\frac{dC^{(0)}}{dt}=-\nu C^{(0)}+\nu p^{(0)}C^{(0)}, \tag{4.2}$$

where

$$p^{(k)}=\frac{(-1)^k}{k!}\int_{-\infty}^{+\infty}p(A)A^k dA,\quad p^{(0)}=1 .$$

The conditions (2.2) will go to

$$C^{(n)}\ 0 = C_0^{(n)} = \frac{(-1)^n}{n!}\int_{-\infty}^{+\infty} W_0(x,t)x^n dx, \quad C^{(0)}(t) \equiv 1 \quad (4.3)$$

Let $\xi_{k,n}(t) = \frac{d^k}{dt^k}C^{(n)}(t)$ (we note that $\xi_{0,n}(0) = C_0^{(n)}$ are known). Then (4.1) can be rewritten in the form:

$$\xi_{k,n} = (-\nu - \alpha n)\xi_{k-1,n} + \nu\sum_{s=0}^{n} p^{(s)}\xi_{k-1,n-s} + \\ +n(n+1)\beta\xi_{k-1,n+1}, \quad n = 1,2,3,\ldots. \quad (4.4)$$

We know the $\xi_{0,n}(0)$, so, believing $t = 0$, you can find $\xi_{1,n}(0)$ from (4.4), knowing $\xi_{1,n}(0)$ you can find $\xi_{2,n}(0)$, etc. By this way, all $\xi_{k,n}(0) = \frac{d^k}{dt^k}C^{(n)}|_{t=0}$ can be found. Consequently, we have the equality

$$C^{(n)}(t) = \sum_{k=0}^{\infty}\frac{\xi_{k,n}(0)}{k!}t^k \text{ for all } n. \quad (4.5)$$

The use of the series (4.5) is more convenient for small $t:\ 0 \le t \le T < 1,$ limited to a finite number of terms. Having constructed a solution on the interval $[0,T]$, and assuming for new initial conditions $C^{(n)}(T)$ you can build $C^{(n)}(t)$ for $t \in [T,2T]$:

$$C^{(n)}(t) = \sum_{k=0}^{\infty}\frac{\xi_{k,n}(0)}{k!}(t-T)^k, t \in [T,2T], \text{ etc.} \quad (4.6)$$

In conclusion, we shall make one useful *remark*. Let

$$\varphi(k,t) = \sum_{n=0}^{\infty}(i)^n C^{(n)}(t)k^n.$$

Then it is easy to see that the following equality holds:

$$W(x,t) = \frac{1}{2\pi}\lim_{\lambda\to+\infty}\int_{-\lambda}^{+\lambda}\varphi(k,t)\, e^{ixk}dk\,. \quad (4.7)$$

## *References*

1. Baranov N.A., Turchak L.I. Numerical solution of the Kolmogorov-Feller equation. – Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki, 2007, vol. 47, N 7, p. 1221-1228 (in Russian).

2. Baranov N.A., Turchak L.I. Numerical solution of the Kolmogorov-Feller equation with singular singularities. – Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki, 2010, vol. 50, N 2, p. 347-351 (in Russian).

3. Aleksandrov V.D. The exact solution of the stationary Kolmogorov-Feller equation. - Surveys on Applied and Industrial Mathematics, 2002, vol. 9, issue 1, p. 106 (in Russian).

4. I.R. King. An Introduction to Classical Stellar Dinamics. - Univ. of California, Berkeley, 1994.

5. Baranov N.A., Turchak L.I. Methods for analysis of functional safety of complex technical systems. - Computing Center of RAS, Moscow, 2006 (in Russian).

6. Firsov A.N. The method of moments in the theory of generalized functions and its applications in problems of system analysis and control. Fundamentals of the theory. - St. Petersburg State Polytechnical University Journal. Computer Science. Telecommunication and Control Systems, issue 6, 2010, p. 74-81 (in Russian).

7. Firsov A.N. Moment representation of rapidly decreasing functions and its applications. - «High intellectual technologies and innovations in education and science»: Materials of XVII International scientific and methodical conference, February 11-12, 2010. Plenary reports. – Saint Petersburg Polytechnic Univ., Saint Petersburg, 2010, p. 114-124 (in Russian).

8. Firsov A.N. Generalized mathematical models and methods for analyzing dynamic processes in distributed systems. – Saint Petersburg Polytechnic Univ., Saint Petersburg, 2013 (in Russian).

9. Firsov A., Koval A. “Moment” representation of “fast decreasing” generalized functions and their application to several applied stochastic problems. - Machines. Technologies. Materials. International journal for science, technics and innovations for the industry (Bulgaria), issue 6, 2015, p. 21-23.