ANALYTIC SOLUTION OF A NONSTATIONARY EQUATION OF KOLMOGOROV-FELLER TYPE WITH A NONLINEAR DRIFT COEFFICIENT

Prof. Dr. Tech. Sci. Andrei N. Firsov
Peter the Great Saint-Petersburg Polytechnic University, Russia
anfirs@yandex.ru

Abstract: The paper is devoted to the construction of analytical algorithms for solving stochastic integro-differential equations of Kolmogorov-Feller type. Such equations are encountered in problems of control theory, communication theory, stellar dynamics, and so on.

Keywords: KOLMOGOROV-FELLER EQUATION, NONLINEAR DRIFT COEFFICIENT, ANALYTICAL SOLUTION ALGORITHMS

1. Introduction

This note is devoted to the construction of an analytical algorithm for solving stochastic integro-differential equations of Kolmogorov-Feller type that occur in problems of control theory, communication theory, stellar dynamics, and so on. In the papers [1 - 5], known to the author and containing variants of analytical and / or numerical algorithms for solving such equations, cases of a linear dependence of the drift coefficient on the spatial coordinate are considered, as a rule. The algorithm of the analytical solution proposed in this paper does not imply such a limitation, and relies on the theory developed earlier by the author of rapidly decreasing generalized functions [6 - 9], where, in particular, the construction of the reconstruction of a "sufficiently fast decreasing function at infinity" is proposed in terms of its power moments. For simplicity of exposition, we restrict ourselves here to the case of a quadratic dependence of the drift coefficient on the spatial coordinate.

2. Formulation of the problem

We seek a solution of the equation

$$\frac{\partial W(x,t)}{\partial t} = \frac{\partial}{\partial x}[(\alpha x + \beta x^2)W(x,t)] + \nu p(A)W(x-A,t)dA - \nu W(x,t)$$

(2.1)

under conditions

$$W(x,0) = W_0(x) \text{ is known function,}$$
$$W(x,t) \rightarrow 0 \text{ for all } t,$$
$$\int_{-\infty}^{\infty} W(x,t) dx = 1$$

(2.2)

with obvious requirements:

$$p(A) \rightarrow 0 \text{ and } \int_{-\infty}^{\infty} p(A) dA = 1.$$

We further assume that

$$p(A) = O(\exp(-|A|^\delta)), \quad |A| \rightarrow \infty$$

(2.3)

for some $\gamma > 0, \quad \delta > 0$.

Solution of the problem (2.1) - (2.2) will be sought in the class of "sufficiently fast" decreasing functions (for details, see below).

3. About the representation of rapidly decreasing functions through its moments

Let $f(x)$ is continuous, and for some $\lambda > 0, \mu > 0$

$$f(x) = O(\exp(-|x|^\mu)), \quad |x| \rightarrow \infty.$$

Let $C^{(n)}(t) = (-1)^n \int_{-\infty}^{\infty} f(x)x^n dx, \quad n = 0,1,2,...$. Using the methods developed in [7-9], it is comparatively easy to prove the relation

$$f(x) = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \sum_{n=0}^{\infty} \frac{C^{(n)}(t)}{\varepsilon^n} \sigma^n \left( \frac{x}{\varepsilon} \right) + O(\varepsilon), \quad \varepsilon \rightarrow 0 .$$

In particular,

$$f(x) = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \sum_{n=0}^{\infty} \frac{C^{(n)}(t)}{\varepsilon^n} \sigma^n \left( \frac{x}{\varepsilon} \right).$$

Here

$$\sigma(x) = \sum_{k=0}^{\infty} (-1)^k \frac{k!}{(k-s)!} \sin \left( \frac{x + (k-s)2\pi}{2} \right).$$

It is not difficult to see that the following equalities hold:

$$\sigma^{(2)}(x) = (-1)^2 \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s}}{(2k + 2s + 1)(2s)!},$$
$$\sigma^{(2k+1)}(x) = (-1)^{k+1} \sum_{s=0}^{\infty} \frac{(-1)^s (k+1)x^{2k+1}}{(2k + 2s + 3)(2s + 1)!},$$
$$\sigma^{(2k)}(x) = O(x^2), \quad x \rightarrow 0,$$
$$\sigma^{(2k+1)}(x) = O(x), \quad x \rightarrow 0,$$
$$\sigma^{(k)}(x) = O(\frac{1}{|x|}), \quad |x| \rightarrow \infty,$$
$$\sigma^{(k)}(x) = \frac{(k)!}{2} \int_{0}^{\frac{1}{|x|}} t^{k-1} e^{-t} dt.$$

4. Construction of the solution of the problem

We put further:

$$C^{(n)}(t) = (-1)^n \int_{-\infty}^{\infty} x^n W(x,t) dx.$$

Then from (2.1) we obtain:

$$\frac{d^k}{dt^k} C^{(n)}(t) = \frac{d^k}{dt^k} \left[ -\nu C^{(n)} + \nu \sum_{s=0}^{\infty} p^{(s)} C^{(n-s)}(t) - \alpha n C^{(n)} \right] +$$
$$+ n(n+1)\beta \frac{d^{k-1}}{dt^{k-1}} C^{(n)}(t), \quad n = 1, 2, 3; \quad k = 1, 2, 3,....$$

$$\frac{dC^{(n)}}{dt} = -\nu C^{(n)} + \nu p^{(n)} C^{(n)},$$

(4.1)

(4.2)

where

$$p^{(k)} = \frac{(-1)^k}{k!} \int_{-\infty}^{\infty} p(A) A^k dA,$$
$$p^{(0)} = 1.$$
The conditions (2.2) will go to

\[ C^{(n)}_0 = C^{(n)}_0 = \left( \frac{(-1)^n}{n!} \int_{-\infty}^{+\infty} W_g(x,t)x^n dx \right) = 1 \]  

\hspace*{1cm} (4.3)

Let \( \xi_{n,s}(t) = \frac{d^n}{dt^n} C^{(n)}(t) \) (we note that \( \xi_{n,s}(0) = C^{(n)}_0 \) are known). Then (4.1) can be rewritten in the form:

\[ \xi_{n,s} = (-\nu - \alpha n)\xi_{n,s-1} + \nu \sum_{i=0}^{n} \frac{\nu^i}{i!} \xi_{n-s,i} + 
+ n(n + 1)\xi_{n+s,n+1}, \quad n = 1, 2, 3, ... \]  

\hspace*{1cm} (4.4)

We know the \( \xi_{n,s}(0) \), so, believing \( t = 0 \), you can find \( \xi_{n,s}(0) \) from (4.4), knowing \( \xi_{n,s}(0) \) you can find \( \xi_{n,s}(0) \), etc. By this way, all \( \xi_{n,s}(0) = \frac{d^n}{dt^n} C^{(n)} \mid_{t=0} \) can be found. Consequently, we have the equality

\[ C^{(n)}(t) = \sum_{k=0}^{n} \frac{\xi_{n,s}(0)}{k!} t^k \]  

\hspace*{1cm} (4.5)

The use of the series (4.5) is more convenient for small \( t: 0 \leq t \leq T \ll 1 \), limited to a finite number of terms. Having constructed a solution on the interval \([0,T]\), and assuming for new initial conditions \( C^{(n)}(T) \) you can build \( C^{(n)}(t) \) for \( t \in [T,2T] \):

\[ C^{(n)}(t) = \sum_{k=0}^{n} \frac{\xi_{n,s}(0)}{k!} (t-T)^k, \quad t \in [T,2T], \text{ etc.} \]  

\hspace*{1cm} (4.6)

In conclusion, we shall make one useful remark. Let

\[ \varphi(k,t) = \sum_{i=1}^{n} (i)^n C^{(n)}(t)k^n. \]

Then it is easy to see that the following equality holds:

\[ W(x,t) = \frac{1}{2\pi} \lim_{\lambda \to +\infty} \int_{-\lambda}^{+\lambda} \varphi(k,t) e^{ikx} dk. \]  

\hspace*{1cm} (4.7)

References


