COMPARISON OF APPROACHES TO ESTIMATION OF TRANSITION MATRIX FOR THE TERRORIST THREAT MARKOV MODEL

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Abstract: *Markovian models are often used in modelling a time development of random phenomena. When modelling real world scenarios it is reasonable to assume that the respective phenomena may not be time homogeneous. Based on the sociological and security research, it can be assumed that there is a link between a destabilisation of a society of a given geographical region and the acts of terrorism. This link is utilised in construction of a model for description of the intensity of a terrorist threat based on given determinants/indicators of societal stability. The model is based on the theory of discrete non-homogeneous Markov chains. The theory of generalised linear models (GLMs) is used in the estimation of the probabilities of the categorised level of the terrorist threat. In the contribution the use of different estimates of the categorised level of terrorist threat probabilities is studied. The estimates are determined by GLMs with different input parameters. The influence of the resulting estimate on the transition matrix of the non-homogeneous Markov chain is assessed. Additionally, a real world example utilising the data from Global Terrorism Database of University of Maryland and Organisation for Economical Cooperation and Development is presented.*
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1 Introduction

In recent years the mathematical models are finding more and more applications in the field of social sciences. Due to current geopolitical events the ability to asses and predict the risk of terrorism has became topical.

The model discussed in this paper aims to describe the evolution of the categorised level of terrorist threat in time. From mathematical viewpoint this might be seen as a time development of random events. In other fields, mathematical models based on the theory of Markov chains are often used to model random events time development, when some categorisation is involved. The markovian framework provides a good compromise between computational feasibility of the model and taking into account the real world dependencies. The finite or countable state space of the chain is particularly suitable to model the considered categories. Such models are often used in economics and finance. Quite common is the application by banking institutions to model the credit risk, i. e. the risk of default (see [1], [2], [3]).

The problematic of modelling of time development of categorised level of terrorist threat bears some resemblance to the credit risk models. For that reason the framework of Markov chains is utilised in this paper as well.

For some particular events a connection to a set of explanatory variables might be found. For example, it is reasonable to believe that the ability of the companies to fulfil their obligations to the credit institution is influenced by the economical development. Thus, it is reasonable to assume a dependence of the level of the terrorist threat on some economical variables, like gross domestic product etc. Some authors have considered models utilising this additional information provided by the covariates (see [4], [5], [6]).

When it comes to modelling the threat of terrorism, a sociological research suggests, that the level of terrorist threat within a given society (country, etc.) may be connected with an overall stability of the studied society (see [7]). There are numerous indicators of stability of a given society. The authors of this paper have already proposed a model utilising this link connecting the terrorist threat with selected indicators (see [8]). The link was used in the estimation of the transition matrix of the Markov chain. In this paper the research is taken a step forward by providing a different and possibly more suitable way of estimation of the transition matrix as well as a comparison of the two approaches.

2 Prerequisites and means for solving the problem

2.1 Markov Chains

The model presented in this paper is based as well as the model presented in [8] on the theory of Markov chains. The formal description does not differ from the one presented in [8]. For the reader's convenience, the authors will recall the key points in this subsection.

A sequence $M = \{M_n : n = 0,1,2,\dots\}$ of random variables, that attains values from a finite set S is called a Markov chain with finite state space S if

$$\text{(1)}\qquad \mathbb{P}[M_{n+1} = l | M_n = k, M_{n-1} = k_{n-1}, \dots, M_1 = k_1, M_0 = k_0] = \mathbb{P}[M_{n+1} = l | M_n = k].$$

for each $l, k, k_{n-1}, \dots, k_0 \in S$. The equation (1) is the well known Markov property.

In the model presented in this paper, the elements $k \in S$ of state space will correspond to the L categories of the level of the terrorist threat.

Out of convenience, henceforth it will not be differentiated between a state of the Markov chain and a category of a level of the terrorist threat. Thus, when we say that the process moved into category k in time step from n to $n+1$, we actually mean that the process jumped into k-th state in time step from n to $n+1$ and so on. Furthermore, from the interpretational point of view, the states (categories) denoted by larger values of integers are considered to be the ones of higher level of terrorist threat (i. e. state (category) $k = 3$ denotes higher level of threat than state (category) $k = 1$).

The transition probabilities are defined as follows

$$\text{(2)}\qquad \pi_{kl}(n, n+1) = \mathbb{P}[M_{n+1} = l | M_n = k],$$

for each $k, l \in S$. The transition probability is the probability the process jumps into state l (the level of the categorised threat changes to l) in the time step from n to $n+1$, provided it was in a state k in time n.

Transition probabilities of a homogeneous Markov chain do not depend on time step n, i. e. $\pi_{kl}(n, n+1) = \pi_{kl}$. In this model, however, it is assumed, that the numbers of terrorist attacks are dependent in each time instant n on a set of m indicator variables $\mathbf{X} = (X_1, \dots, X_m)$. These variables vary in time step n, and hence the notation $\mathbf{X} = \mathbf{X}_n = (X_{n1}, \dots, X_{nm})$ is used. Thus, the resulting Markov chain is non-homogeneous. The transition probabilities of the Markov chain with L states are collected in the transition matrix $\mathbf{P}_{n,n+1}(\mathbf{X}_n)$ of a type $L \times L$ given by

$$\text{(3)}\mathbf{P}_{n,n+1}(\mathbf{X}_n) = \begin{pmatrix} \pi_{11}(n, n+1; \mathbf{X}_n) & \dots & \pi_{1L}(n, n+1; \mathbf{X}_n) \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \pi_{L1}(n, n+1; \mathbf{X}_n) & \dots & \pi_{LL}(n, n+1; \mathbf{X}_n) \end{pmatrix}.$$

The notation $\mathbf{P}_{n,n+1}(\mathbf{X}_n)$ is used to further stress out the fact, that the transition matrix depends on the indicator variables.

The properties of the transition matrix are discussed more in detail in [8] or any classical book on Markov theory (e. g. [9]).

Given the state vector $\mathbf{p}_0 = (p_{1,0}, \dots, p_{L,0})$ in time 0, where $p_{k,0} = \mathbb{P}[M_0 = k]$ for $k \in S$, it is possible to obtain a state vector $\mathbf{p}_n = (p_{1,n}, \dots, p_{L,n})$, where $p_{k,0} = \mathbb{P}[M_n = k]$ for some $n \in \mathbb{N}_0$ in the following way

(4) $$\mathbf{p}_n = \mathbf{p}_0 \mathbf{P}_{0,1}(\mathbf{X}_1)\mathbf{P}_{1,2}(\mathbf{X}_2)\dots\mathbf{P}_{n-2,n-1}(\mathbf{X}_{n-1})\mathbf{P}_{n-1,n}(\mathbf{X}_n).$$

The transition matrices $\mathbf{P}_{0,1}(\mathbf{X}_1), \dots, \mathbf{P}_{n-1,n}(\mathbf{X}_n)$ need to be estimated.

2.2 Multinomial Response Generalised Linear Models

Generalised linear model (GLM) is used to connect the categorised level of terrorist threat with the indicator variables. The GLM is used to obtain an estimate of the transition matrix row by row. In the paper [8], a Poisson GLM is utilised. Here the use of the multinomial response generalised linear models is proposed (see [10], [11]).

Let M_n for $n \in \mathbb{N}$ be a nominal scale response variable with L categories. The multicategorial model used in this paper to obtain the estimates of the transition probabilities in k-th category is the multinomial logit model (see [10], [11]), that is given by

(5) $$\log\left(\frac{\mathbb{P}[M_n = l]}{\mathbb{P}[M_n = L]}\right) = \beta_{l0,k} + \mathbf{x}_{n\bullet,k}\boldsymbol{\beta}_{l,k},$$

where $\beta_{l0,k}$ and $\boldsymbol{\beta}_{l,k} = (\beta_{l1,k}, \dots, \beta_{l(L-1),k})$ are unknown parameters, which correspond to state $k = 1, \dots, L-1$. For the baseline category $l = L$, set $\beta_{L0,k} = 0$ and $\boldsymbol{\beta}_{L,k} = \mathbf{0}$. The $\mathbf{x}_{n\bullet}$ are the values of the indicator variables $\mathbf{X}_n$ at the time step n.

3 *Solution of the examined problem*

The multinomial logit GLM was used repeatedly in sequence for each state $k \in \{1, \dots, L\}$ the process visited. In this way a vector $\hat{\pi}_{k\bullet} = (\hat{\pi}_{k1}, \dots, \hat{\pi}_{kL})$ of probability estimates was obtained, such that the respective probability estimates correspond to the k-th row of the desired transition matrix estimate.

3.1 Transition Matrix Estimation Using Multinomial Logit GLM

The multinomial logit GLM described in the Subsection 2.2 is computed for each category $k \in \{1, \dots, L\}$. Hence, for each category $k \in \{1, \dots, L\}$, the estimate $\widehat{\boldsymbol{\beta}}_{\boldsymbol{k}} = \left(\hat{\beta}_{10,k}, \widehat{\boldsymbol{\beta}}^{\boldsymbol{T}}_{\boldsymbol{1,k}}, \dots, \hat{\beta}_{L0,k}, \widehat{\boldsymbol{\beta}}^{\boldsymbol{T}}_{\boldsymbol{L,k}}\right)$ of the vector of parameters of the GLM is obtained.

For the multinomial logit GLM for each category $k \in \{1, \dots, L\}$ the estimates of $L-1$ fitted values $\hat{\pi}_{k1}(\mathbf{x}_{n\bullet}), \dots, \hat{\pi}_{k(L-1)}(\mathbf{x}_{n\bullet})$, given the vector of covariates $\mathbf{x}_{n\bullet} = (x_{n1}, \dots, x_{nm})$ for some $n \in \mathbb{N}$ are obtained through a formula

(6) $$\hat{\pi}_{kl}(\mathbf{x}_{n\bullet}) = \frac{\exp\left(\hat{\beta}_{l0} + \mathbf{x}_{n\bullet}\hat{\boldsymbol{\beta}}_l\right)}{1 + \sum_{r=1}^{L} \exp\left(\hat{\beta}_{r0} + \mathbf{x}_{n\bullet}\hat{\boldsymbol{\beta}}_r\right)},$$

where $l = 1, \dots, L-1$. For $l = L$ we then have

(7) $$\hat{\pi}_{kL}(\mathbf{x}_{n\bullet}) = 1 - \sum_{r=1}^{L-1} \hat{\pi}_{kr}.$$

The estimate of the probability of transition from state k to state l in time instant from n to $n+1$ for $n \in \{1, \dots, s\}$ is then defined as $\hat{\pi}_{kl}(n, n+1; \mathbf{X}_n) = \hat{\pi}_{kl}(\mathbf{x}_{(n+1)\bullet})$.

3.2 Transition Matrix Estimation Using Poisson GLM

The detailed description of the method of obtaining the estimates of the transition matrix using the Poisson GLM is provided in the paper [8]. The key points will be recalled here.

The response in this case are the numbers of terrorist attacks. These are split into categories together with the corresponding observations of explanatory variables, based on whether the value of the observation of the response lies between a selected bounds of the respected category.

The Poisson GLM is given by

(8) $$\log(\lambda_n) = \beta_0 + \mathbf{x}_{n\bullet}\boldsymbol{\beta},$$

where the expectation λ_n (the average number of terrorists attacs in the given category) depends on unknown parameters β_0, $\boldsymbol{\beta}$ and on the values $\mathbf{x}_{n\bullet}$ of the indicator indicator variables $\mathbf{X}_n$ at the time step n.

This model is again applied for each category $k = 1, \dots, L$ in order to obtain vector of estimates $\hat{\alpha}_k = (\hat{\beta}_{0,k}, \hat{\boldsymbol{\beta}}^{\boldsymbol{T}}_{\boldsymbol{k}})$ of the parameters $\alpha_k = (\beta_{0,k}, \boldsymbol{\beta}^{\boldsymbol{T}}_{\boldsymbol{k}})$.

The estimate $\hat{\lambda}_{nk}$ of the parameter λ_{nk} for each category $k = 1, \dots, L$ is then obtained via the formula

(9) $$\hat{\lambda}_{nk} = e^{\hat{\beta}_{0,k} + \mathbf{x}_{n\bullet}\hat{\boldsymbol{\beta}}_{\boldsymbol{k}}}.$$

Then a random variable Z_k with Poisson distribution with the value $\hat{\lambda}_k$ as an expectation parameter is defined for each category $k = 1, \dots, L$. The transition transition from a category k into a category l, $k, l \in S$ is then defined as the probability of the random variable Z_k attaining values between the lower and upper bound of the category.

4 *Results and discussion*

4.1 Real Data Example

The approaches proposed in this paper and in paper [8] were applied to estimate a transition matrix using a real dataset. The dataset consisted of quarterly numbers of terrorist attacks in France between the years 1981 and 2018 obtained from the Global Terrorism Database of University of Maryland, and two indicator explanatory variables, the gross domestic product and the unemployment rate obtained from the Organisation for Economical Development. Detailed description of the dataset is provided in [8]. For the readers convenience the authors will present figures of the dataset in question.

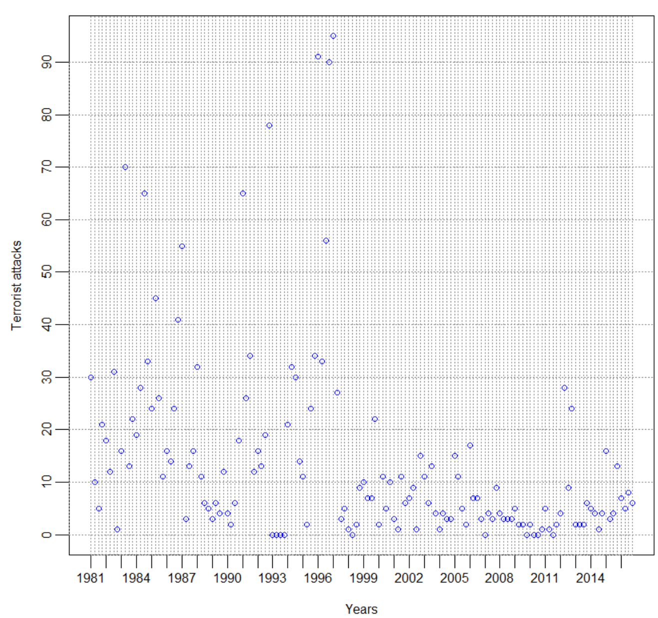

Fig. 1: *Time series of quarterly observations of the numbers of terrorist attacks*

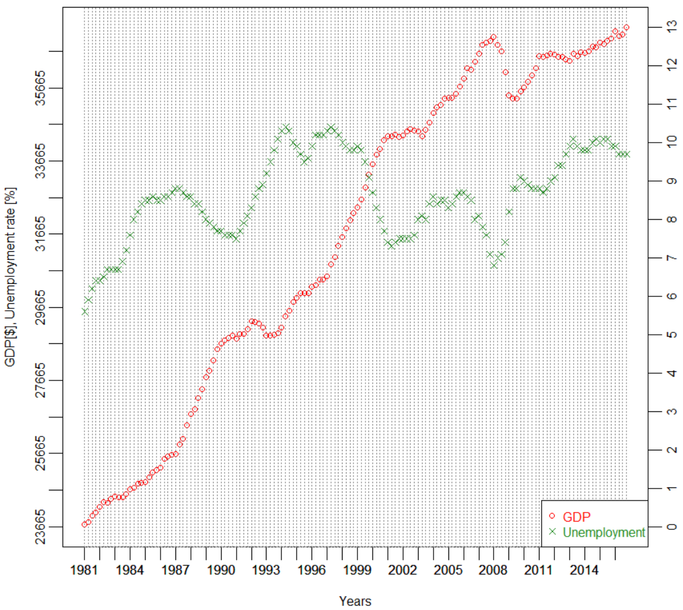

Fig. 2: Time series of quarterly observations of GDP in constant prices of US Dollar (left horizontal axis) and of unemployment rate in percentage of labour force (right horizontal axis)

From the Fig 2 one can see that, the gross domestic product grew in average over time, while the unemployment rate stagnated or increased slightly. This development might be considered still more or less optimistic scenario. It should be noted, that there might be a relation between the two explanatory variables (see Fig 2). The last available observation of the explanatory variables in the year 2016 was taken as the "future" observation.

Categorisation of the data was carried out in the following manner. Into the first category were placed observations greater or equal to 0 and lesser than 6, into the second observations greater or equal to 6 and lesser than 13. Into the third category were placed observations greater or equal to 13. The transition matrix estimate obtained using the Poisson GLM described in [8] is the following:

$$(10) \qquad \widehat{\mathbf{P}}_{0,1;Po}(\mathbf{X}) = \begin{pmatrix} 0.990 & 0.010 & 0.000 \\ 0.335 & 0.636 & 0.029 \\ 0.000 & 0.007 & 0.993 \end{pmatrix}.$$

The matrix obtained using the multinomial logit GLM described in this paper is

$$(11) \qquad \widehat{\mathbf{P}}_{0,1;Mult}(\mathbf{X}) = \begin{pmatrix} 0.812 & 0.095 & 0.093 \\ 0.361 & 0.530 & 0.109 \\ 0.457 & 0.453 & 0.090 \end{pmatrix}.$$

The transition matrix estimated using the multinomial logit GLM reflects the optimistic development of responses, favouring the category with the least terrorist threat. This is not entirely true about the transition matrix obtained via the Poisson GLM described in [8]. In the estimate (10) the probabilities $\hat{\pi}_{kk}(0,1)$ on the diagonal attain large values, while the further we move in the row away from the diagonal entry, the lower the estimated probability (compare $\hat{\pi}_{12}(0,1)$ and $\hat{\pi}_{13}(0,1)$ in (10). Additionally, the probability $\hat{\pi}_{33}(0,1)$ of staying in the third category of the highest terrorist threat seems to be unreasonably high, given the optimistic scenario. In [8] it was suggested, that this might rather be due to the properties of the Poisson GLM itself, and does not reflect the reality very well.

Note, that the size of real input dataset is rather small, in addition the observations of terrorist attacks contain a large number of very small observations and a small set of relatively high values (see Fig. 1). This, altogether with the particular choice of the bounds of the categories further worsens the effect that the Poisson model has on the estimates of the entries on the diagonal.

The problem of unreasonably high values on the diagonal seems not to be present in the estimate (11). The probability is more evenly distributed among the respective categories, while at the same time reflecting the positive development scenario. For the first and the third row we have $\hat{\pi}_{kl} > \hat{\pi}_{kr}$ for k fixed and $l < r$. The exception is the second row, where the probability jumping from the second state back to the second state is the highest.

The link between the response observations and the covariates was weak when modelled via the Poisson GLM. Only for the category $k = 3$ the tests for each parameter rejected the null hypothesis $H_0: \hat{\beta}_{j,k} = 0$ for $j = 0,1,2$. In the case of multinomial logit GLM the link between the response and the covariates was weak as well. Additionally, no matter the choice of the baseline category, a separation occurred for some logits (see [10]), rendering the tests of significance for the respective parameters impossible.

5 Conclusion

The paper [8] presents a non-homogeneous Markov chain model for prediction of a categorised level of terrorist threat. The non-homogeneity of the Markov chain stems from the assumption that the categorised level of the terrorist threat and thus the transition matrix of the chain is dependent on a set of indicators of destabilisation of a society, that vary in time. A Poisson GLM is used to connect the categorised level of terrorist threat with the indicator covariates and estimate the transition matrix row by row. This paper proposes a modified approach using multinomial logit GLM instead of the Poisson one. A comparison of the two approaches is carried out on a real dataset.

The estimates of the transition probabilities on the diagonal of the transition matrix obtained via the Poisson GLM tend to attain large values. This is assumed to be rather a property of the Poisson GLM, that does not reflect the real situation. Especially unexpected is the large value of the probability of staying in the category of the largest terrorist threat in case of an optimistic development scenario. This effect is not present, when the multinomial logit GLM is applied instead. The transition probabilities tend to be more evenly distributed among the respective categories, while reflecting the character of the scenario (for the rather optimistic scenario given by the real data the transition to the categories of lower threat tend to be higher than the ones to the categories of higher threat).

It should be noted, however, that in general the connection between the responses and the covariates regardless of the used GLMs was weak. Additionally, the size of the currently available real data sample proved to be insufficient. The proposed multinomial logit model did not share the diagonal values-increasing effect of the Poisson model, and thus seems to be more suitable, in further research however, it would be necessary to choose different set of indicators to further study the model. Additionaly, a way to generate the datasets for a thorough simulations study would need to be developed. In order to increase the sample size a panel data approach may be considered. That is however, beyond the scope of this paper.

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