SUBSTANTIATION OF J.-G. SUN'S HYPOTHESIS, WHICH LIES IN THE BASIS OF THE THEORY OF ANALYTICAL DEPENDENCE OF EIGENVALUES OF MATRIX FROM "DISTURBING" PARAMETERS UNDER MULTI-PARAMETRIC PERTURBATION OF THE MATRIX ELEMENTS

ОБОСНОВАНИЕ ГИПОТЕЗЫ J.-G. SUN'A, ЛЕЖАЩЕЙ В ОСНОВЕ ТЕОРИИ АНАЛИТИЧЕСКОЙ ЗАВИСИМОСТИ СОБСТВЕННЫХ ЗНАЧЕНИЙ МАТРИЦЫ ОТ "ВОЗМУЩАЮЩИХ" ПАРАМЕТРОВ ПРИ МНОГОПАРАМЕТРИЧЕСКОМ ВОЗМУЩЕНИИ МАТРИЧНЫХ ЭЛЕМЕНТОВ

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Abstract: *The paper gives a rigorous justification of Ji-Guang Sun's hypothesis about the properties of the eigenvalues of the matrix of a linear dynamical system under multiparametric perturbation of its elements.*
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1. Introduction

The concept of stability of dynamic systems characterizes the property of a system to operate stably in modes where there are uncertainties in the values of certain system elements. Due to the complexity, and sometimes the practical impossibility to indicate the necessary and sufficient allowable ranges of variation of the corresponding parameters, at least sufficient estimates may be of great interest. On the other hand, practice shows that the desire for universality of theoretical results, as a rule, leads to great difficulties in the application of such results for solving specific problems. We believe that the consideration of such considerations should underlie the construction of theoretical structures aimed at solving specific applied problems. In this paper, we propose a solution to the problem of conditions sufficient to preserve the stability property of a linear dynamical system under small perturbations of its matrix. In this case, an estimate of the smallness of the perturbation parameters is given.

2. Formulation of the problem

Let's consider a dynamic system:

$$\frac{dX(t)}{dt} = AX(t)$$
$$A = (a_{ij})_{i,j=1}^{n},\ X(t) = (x_1(t), x_2(t), ..., x_n(t))^T. \qquad (2.1)$$

Here A denotes an "unperturbed" matrix. Suppose the matrix A has simple different eigenvalues $\lambda_j,\ j = 1, 2, ..., n$ and $\operatorname{Re}(\lambda_j) < 0,\ j = 1, 2, ..., n,$ that is, the system (2.1) is supposed to be sustainable.

Let further

$$\vec{\varepsilon} = (\varepsilon_{11}, \ldots, \varepsilon_{1n}, \varepsilon_{21}, \ldots, \varepsilon_{2n}, \ldots, \varepsilon_{nn})^T \in \mathbb{R}^{n^2},$$

where ε_{ij} – unknown "perturbations" of the elements of the original matrix. Let's consider perturbation parameters ε_{ij} will be small enough in the sense that the quantity ε_{ij}^2 can be neglected in comparison with ε_{ij}: $\varepsilon_{ij}^2 \ll |\varepsilon_{ij}|$. The ultimate goal is a "sufficient" estimate of the magnitude of the elements ε_{ij} of a "perturbed" system

$$\frac{dX(t)}{dt} = A(\vec{\varepsilon})\, X(t), \qquad (2.2)$$

under which the stability property of its solutions is preserved.

Further we will consider that

$$A(\varepsilon_{11}, \varepsilon_{12}, \ldots, \varepsilon_{1n}, \varepsilon_{21}, \ldots, \varepsilon_{2n}, \ldots, \varepsilon_{nn}) = A + \sum_{i,j=1}^{n} \varepsilon_{ij} A_{ij}, \qquad (2.3)$$

where A_{ij} – are known matrices. Ji-Guang Sun has proved [1], that in the case of the validity of "Sun's hypothesis" (see (3.1), (3.2) below), eigenvalues $\lambda_j(\vec{\varepsilon}),\ j = 1, 2, ..., n$ of a perturbed matrix $A(\vec{\varepsilon})$ are fairly smooth functions of parameters $\varepsilon_{ij},\ i, j = 1, 2, ..., n$ in the neighborhood of zero. This allows, if we assume the validity of the Sun's hypothesis, to answer the question raised above about the stability conditions of a perturbed system (2.2) (see [2]). In this paper we will show that the "hypothesis" (3.1), (3.2) is in fact a theorem.

Under the norms of matrices and vectors, we will further understand the corresponding Euclidean norms.

3. Proof of Sun's hypothesis

The main results on the dependence of eigenvalues on perturbing parameters were obtained by T. Kato [3] (for one parameter) and by J.-G. Sun [1] (for several parameters). However, when formulating the results, J.-G. Sun expressed a hypothesis about the nature of the dependence of the eigenvalues on the perturbation parameters, which he did not prove. In this paper, we justify this hypothesis.

In [1] J.-G. Sun stated the following hypothesis, on which the proof of the main theorem in the paper [1] was based:

Sun's hypothesis. Let λ_s – non-multiple eigenvalue, generally speaking, asymmetric matrix $A \in \mathbb{R}^{n\times n}$, $\vec{x}_s$ and $\vec{y}_s$ – corresponding right and left eigenvectors, at the same time we accept, that $\|\vec{x}_s\| = 1$ (where $\|\vec{x}_s\| = \sqrt{\sum_{i=1}^{n} (x_{si})^2}$, $x_{si}, i = 1, \ldots, n$ are coordinates of the vector $\vec{x}_s$) and $\vec{y}_s^T \cdot \vec{x}_s = 1$. Then for any λ_s there exist such $\tilde{X}, \tilde{Y} \in \mathbb{R}^{n\times(n-1)}$, that for matrices $X = (\vec{x}_s, \tilde{X})$ and $Y = (\mathrm{y}_s, \tilde{Y})$, the following relations hold:

$$Y^T X = I_n, \qquad (3.1)$$

$$Y^T A X = \begin{pmatrix} \lambda_s & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & \tilde{A} \end{pmatrix},\ \lambda_s \notin \lambda(\tilde{A}). \qquad (3.2)$$

where $\lambda_s,\ s = \overrightarrow{1, n}$ – are non-multiple matrix eigenvalues A.

Here we give a detailed *proof* of this result. Let us consider the matrix

$$Z' = \left(\vec{x}_s \left| \begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{matrix} \right. \right)_{n\times n} = \left(\begin{matrix} x_{s1} \\ x_{s2} \\ \cdots \\ x_{s(n-1)} \\ x_{sn} \end{matrix} \left| \begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{matrix} \right. \right)_{n\times n}.$$

Applying the Gram-Schmidt orthogonalization (see, for example [4, 5]) to the matrix Z', we obtain the matrix $Z = \left(\vec{x}_s, \tilde{Y}\right)-$ orthonormal in the columns, i.e.

$$\tilde{Y}^T \vec{x}_s = 0_{(n-1)\times 1}. \qquad (3.3)$$

It is also easy to check that $\det Y = \left(y_s, \tilde{Y}\right) \neq 0$.

Let $H = \left\{h_{i,j}\right\}_{i=1,\, j=1}^{n} \equiv Y^{-1} = \left(y_s, \tilde{Y}\right)^{-1}$, and consider the matrix $X = \left(\vec{x}_s, \tilde{X}\right)$, were

$$\tilde{X}_{(n\times(n-1))} = \left[\left\{h_{i,j}\right\}_{i=2,\, j=1}^{n}\right]^T = \left[\left(h_2\right)^T \quad \left(h_3\right)^T \quad \cdots \quad \left(h_n\right)^T\right],$$

were h_i, $i = 2 \ldots n-$ are rows of the matrix H, i.e.

$$\vec{y}_s^{\,T} \tilde{X} = 0_{1\times(n-1)} \text{ and } \tilde{Y}^T \tilde{X} = I_{(n-1)}. \qquad (3.4)$$

From (3.3) and (3.4) we have:

$$Y^T X = \left(\vec{y}_s, \tilde{Y}\right)^T \left(\vec{x}_s, \tilde{X}\right) = \begin{pmatrix} \vec{y}_s^{\,T} \vec{x}_s & \vec{y}_s^{\,T} \tilde{X} \\ \tilde{Y}^T \vec{x}_s & I_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & I_{n-1} \end{pmatrix} = I_n,$$

therefore, we have a relationship (3.1).

Consider further:

$$Y^T A = (\vec{y}_s, Y_2, Y_3, \ldots, Y_n)^T (\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n) =$$

$$= \begin{pmatrix} \vec{y}_s^{\,T} \vec{a}_1 & \vec{y}_s^{\,T} \vec{a}_2 & \ldots & \vec{y}_s^{\,T} \vec{a}_n \\ & A^* & & \end{pmatrix} = \begin{pmatrix} \vec{y}_s^{\,T} A \\ A^* \end{pmatrix} = \begin{pmatrix} \lambda_s \vec{y}_s^{\,T} \\ A^* \end{pmatrix}, \qquad (3.5)$$

where $\vec{a}_j$, $j = 1, 2, \ldots, n$ are matrix columns of the matrix *A*, and matrix $A^* \in \mathbb{R}^{(n-1)\times n}$. Y_j, $j = 2, 3, \ldots, n$ - are matrix columns of Y.

Next, multiply the expression (3.5) by the matrix X on the right:

$$Y^T A X = \begin{pmatrix} \lambda_s \vec{y}_s^{\,T} \\ A^* \end{pmatrix} (\vec{x}_s, X_2, X_3, \ldots, X_n) =$$

$$= \begin{pmatrix} \lambda_s \vec{y}_s^{\,T} \vec{x}_s & \lambda_s \vec{y}_s^{\,T} X_2 & \ldots & \lambda_s \vec{y}_s^{\,T} X_n \\ & & A^{**} & \end{pmatrix} = \qquad (3.6)$$

$$= \begin{pmatrix} \lambda_s & 0_{1\times n-1} \\ & A^{**} \end{pmatrix},$$

where X_j, $j = 2, 3, \ldots, n$ - are matrix columns of *X*, and matrix $A^{**} \in \mathbb{R}^{(n-1)\times n}$.

Next we have for AX:

$$AX = \begin{pmatrix} \vec{a}^1 \\ \vec{a}^2 \\ \vdots \\ \vec{a}^n \end{pmatrix} \cdot \left(\vec{x}_s, X_2, X_3, \ldots, X_n\right) = \left(\lambda_s \vec{x}_s, \quad A'\right), \qquad (3.7)$$

where $\vec{a}^i$ - are rows of matrix A; X_j, $j = 2, 3, \ldots, n$ - are columns of matrix *X*, and matrix $A' \in \mathbb{R}^{n\times(n-1)}$.

Next, multiply the expression (3.7) by Y^T on the left:

$$Y^T A X = (\vec{y}_s, Y_2, Y_3, \ldots, Y_n)^T \cdot \left(\lambda_s \vec{x}_s, \quad A'\right)$$

$$= \begin{pmatrix} \lambda_s & \\ 0_{(n-1)\times 1} & A'' \end{pmatrix}, \qquad (3.8)$$

where Y_j, $j = 2, 3, \ldots, n$ - are the columns of matrix *Y*, and matrix $A'' \in \mathbb{R}^{n\times(n-1)}$.

From the property of matrix associativity, and taking into account expressions (3.6), (3.8), we are convinced of the validity of equality (3.2).

We now give an example of the application of the results.

Let the matrix *A* have three different negative eigenvalues $\lambda_1, \lambda_2, \lambda_3$.

Let $\vec{x}_1 = \left(x_{11} \quad x_{12} \quad x_{13}\right)^T$ and $\vec{y}_1 = \left(y_{11} \quad y_{12} \quad y_{13}\right)^T$ - right and left eigenvectors corresponding to λ_1. At the same time, the eigenvectors satisfy the following conditions:

$$\left\|\vec{x}_1\right\| = 1 \text{ и } \vec{y}_1^T \vec{x}_1 = 1.$$

The case $x_{11} \neq \pm 1$.

Let construct matrices Y, X in this case (see above). Let

$$Z' = \left(\vec{x}_1 \left| \begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix} \right. \right) \equiv \left(\begin{matrix} x_{11} \\ x_{12} \\ x_{13} \end{matrix} \left| \begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix} \right. \right), \quad \vec{x}_1 = \begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \end{pmatrix}.$$

Applying Gram-Schmidt orthogonalization to the matrix Z' we obtain the matrix orthonormal in the columns

$$Z = \left(\vec{x}_1, \tilde{Y}\right) = \begin{pmatrix} x_{11} & \dfrac{1-x_{11}^2}{\sqrt{1-x_{11}^2}} & 0 \\ x_{12} & \dfrac{-x_{11}x_{12}}{\sqrt{1-x_{11}^2}} & \dfrac{x_{13}}{\sqrt{1-x_{11}^2}} \\ x_{13} & \dfrac{-x_{11}x_{13}}{\sqrt{1-x_{11}^2}} & \dfrac{-x_{12}}{\sqrt{1-x_{11}^2}} \end{pmatrix},$$

with the condition

$$x_{11} \neq \pm 1. \qquad (3.9)$$

Similarly, in accordance with the above, we obtain the matrix $Y = \left(\vec{y}_1, \tilde{Y}\right)$:

$$Y = \begin{pmatrix} y_{11} & \dfrac{1-x_{11}^2}{\sqrt{1-x_{11}^2}} & 0 \\ y_{12} & \dfrac{-x_{11}x_{12}}{\sqrt{1-x_{11}^2}} & \dfrac{x_{13}}{\sqrt{1-x_{11}^2}} \\ y_{13} & \dfrac{-x_{11}x_{13}}{\sqrt{1-x_{11}^2}} & \dfrac{-x_{12}}{\sqrt{1-x_{11}^2}} \end{pmatrix}. \qquad (3.10)$$

Its determinant is equal to

$$\det(Y) = \frac{\left(1-x_{11}^2\right)\left(x_{11}y_{11} + x_{12}y_{12} + x_{13}y_{13}\right)}{\left(1-x_{11}^2\right)} = 1,$$

whence it follows that there exists an inverse matrix for *Y*:

$$H = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ \dfrac{x_{12}y_{12} + x_{13}y_{13}}{\sqrt{1-x_{11}^2}} & \dfrac{-y_{11}x_{12}}{\sqrt{1-x_{11}^2}} & \dfrac{-x_{13}y_{11}}{\sqrt{1-x_{11}^2}} \\ \dfrac{x_{11}\left(x_{12}y_{13} - x_{13}y_{12}\right)}{\sqrt{1-x_{11}^2}} & -\dfrac{x_{11}^2 y_{13} - x_{11}y_{11}x_{13} - y_{13}}{\sqrt{1-x_{11}^2}} & \dfrac{x_{11}^2 y_{12} - x_{11}y_{11}x_{12} - y_{12}}{\sqrt{1-x_{11}^2}} \end{pmatrix}$$

Next, we build the matrix $X = \left(\vec{x}_1, (h_2)^T, (h_3)^T\right)$, where $h_2, h_3 -$ rows of matrix H, i.e.

$$X = \left(\vec{x}_1, \tilde{X}\right) = \begin{pmatrix} x_{11} & \frac{x_{12}y_{12} + x_{13}y_{13}}{\sqrt{1-x_{11}^2}} & \frac{x_{11}\left(x_{12}y_{13} - x_{13}y_{12}\right)}{\sqrt{1-x_{11}^2}} \\ x_{12} & \frac{-y_{11}x_{12}}{\sqrt{1-x_{11}^2}} & -\frac{x_{11}^2 y_{13} - x_{11}y_{11}x_{13} - y_{13}}{\sqrt{1-x_1^2}} \\ x_{13} & \frac{-x_{13}y_{11}}{\sqrt{1-x_{11}^2}} & \frac{x_{11}^2 y_{12} - x_{11}y_{11}x_{12} - y_{12}}{\sqrt{1-x_{11}^2}} \end{pmatrix}, \quad (3.11)$$

$$\det(X) = 1$$

It is not difficult to see that $X = \left(Y^{-1}\right)^T$, so $Y^T X = I$, i.e. (3.1) is right.

From (3.10) and (3.11) we finally get:

$$Y^T AX = \begin{pmatrix} \lambda_1 \vec{y}_1^T \vec{x}_1 & 0 & 0 \\ 0 & \tilde{a}_{22} & \tilde{a}_{23} \\ 0 & \tilde{a}_{32} & \tilde{a}_{33} \end{pmatrix} \quad (3.12)$$

and $\tilde{A} = \begin{pmatrix} \tilde{a}_{22} & \tilde{a}_{23} \\ \tilde{a}_{32} & \tilde{a}_{33} \end{pmatrix}$,

as required.

The case of $x_{11} = \pm 1$ is considered by direct calculation.

4. Conclusions

The paper presents a proof of the hypothesis of J.-G. Sun, which is the basis of the theorem on the analytical properties of the eigenvalues of a matrix in the case of multiparameter perturbation of its elements.

5. Literature

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