INVESTIGATION OF STRENGTH OF THICK-WALLED CIRCULAR CYLINDER BY USING BOUNDARY VALUE PROBLEMS OF ELASTICITY

PhD. Zirakashvili Natela
I. Vekua Institute of Applied Mathematics of Iv. Javakhishvili Tbilisi State University, Georgia
natzira@yahoo.com

Abstract: *The strength of a sufficiently long thick-walled homogeneous isotropic circular tube (cylinder) under the action of external forces is studied using the problems of elasticity statics. In particular, there are established the minimum thickness of pipes with different materials and with different diameters, for which do not exceed the permissible stresses values. Cylinder is in state of plane deformation, therefore are considered a two-dimensional boundary value problems for circular ring. Represented tables and graphs of minimum thickness of a circular ring when a) the normal constant stresses act at internal border, while the outer boundary is free of stresses and b) the normal constant stresses act at external border, while the inner boundary is free of stresses. To the numerical realization above mentioned problems are used solutions obtained by two means: the analytical solution obtained by method of separation of variables, and Lame's solution.*

Keywords: THICK-WALLED PIPE, BOUNDARY VALUE PROBLEMS ELASTICITY, POLAR COORDINATES, SEPARATION OF VARIABLES METHOD

1. Introduction

Thick-walled pipes (cylinders) are often used in many different branches of industry and they are often under internal pressure or external load. By varying the pipe parameters (diameter, wall thickness and material (elastic properties)), the optimal values of parameters, owing to which the pipe does not disintegrate (is not cracked), will be selected.

Many scientists have considered different problems for a thick-walled cylinder [1-4]. Article [1] studies the behavior of rotating thick-walled cylinders made of rubber-like materials. In work [2], the authors consider the influence of the parameters of geometrical shape and properties of the materials on the limit load of a thick-walled cylinder. High internal pressures in a thick-walled cylinder produce great stresses on the internal surface of the cylinder. Therefore, the analysis of a thick-walled cylinder under the impact of high internal pressure, in particular, the identification of the stress concentration coefficient in a cylinder with and without pressure, is topical [3]. Work [4] studies the influence of pressure and deformation on creep stresses in a thick-walled cylinder, which is made up of functionally graded material and is subjected to torsion.

The given work studies the dependence of the strength of quite a long thick-walled circular cylinder on the wall thickness and material. In particular, the minimum thicknesses of the walls of the cylinders made up of various homogenous isotropic materials and of different diameters in the state of plane deformation, with which the stresses in the cylinders do not exceed the admissible values, is identified.

In order to solve the boundary-value and boundary-contact problems for the areas with a curvilinear boundary, it is expedient to consider these problems in a relevant curvilinear coordinate system. For example, the problems for the areas bordered with a circle or its parts are considered in a polar coordinate system [5-8]; for the areas bordered with an ellipse, hyperbola or their parts, the problems are considered in an elliptic coordinate system [9-13]; for the areas bordered with a parabola or its parts, they are considered in a parabolic coordinate system [14-16], while for the areas bordered with eccentric circles, the problems are considered in a bipolar coordinate system [17-19]. Thus, the boundary-value problems for a circular cylinder in the state of plane deformation are considered in a polar coordinate system.

2. Principal equalities in polar coordinates and posing boundary value problems

In case of absence of volume forces, a system of equilibrium equations written in polar coordinates is obtained as a result of projecting a known differential equation of the equilibrium of a homogenous elastic isotropic body [21] $(\lambda+2\mu)\text{graddiv}\,\vec{U}-\mu\,\text{rotrot}\vec{U}=0$ on the coordinate axes of polar $r,\ \alpha$ coordinate system. The given system of equilibrium equations can be written down by using functions K, B, u, v as follows:

$$\begin{aligned}&(a)\ \ rK_{,r}-B_{,\alpha}=0, \quad (c)\ \ u_{,r}+\frac{u}{r}+\frac{1}{r}\mathrm{v}_{,\alpha}=\frac{K}{\lambda+2\mu}\\ &(b)\ \ rB_{,r}+K_{,\alpha}=0, \quad (d)\ \ \mathrm{v}_{,r}+\frac{\mathrm{v}}{r}-\frac{1}{r}u_{,\alpha}=\frac{B}{\mu}.\end{aligned} \qquad (1)$$

Here $B=B_z=\text{rot}_z\vec{U}=\mathrm{v}_{,r}-\frac{1}{r}u_{,\alpha}+\frac{\mathrm{v}}{r}$, $K=\text{div}\vec{U}=u_{,r}+\frac{u}{r}+\frac{1}{r}\mathrm{v}_{,\alpha}$.

Thus, we have a plane deformed state when there remain only radial $u(r,\alpha)$ and circular $\mathrm{v}(r,\alpha)$ components of three components of vector $\vec{\mathrm{U}}$ and normal stresses R_r, A_α and tangential (shearing) stress R_α of the stress tensor components.

The stresses are expressed with the following equations by means of displacements [20]:

$$\begin{aligned}&R_r=\lambda\left(u_{,r}+\frac{u}{r}+\frac{1}{r}\mathrm{v}_{,\alpha}\right)+2\mu\,u_{,r},\\ &A_\alpha=\lambda\left(u_{,r}+\frac{u}{r}+\frac{1}{r}\mathrm{v}_{,\alpha}\right)+2\mu\left(\frac{1}{\mathrm{r}}\mathrm{v}_{,\alpha}+\frac{u}{r}\right),\\ &R_\alpha=\mu\left(\mathrm{v}_{,r}-\frac{\mathrm{v}}{r}+\frac{1}{r}u_{,\alpha}\right).\end{aligned} \qquad (2)$$

Here $\lambda=\frac{\nu E}{(1+\nu)(1-2\nu)}$, $\mu=\frac{E}{2(1+\nu)}$, where E is modulus of elasticity and ν is Poisson's ratio.

By virtue of the symmetry principle, finally the problem will be posed for one-fourth of circular ring $\tilde{\Omega}$: $\Omega=\left\{r_1<r<r_2, 0<\alpha<\frac{\pi}{2}\right\}$. Due to the plane deformation, there is no third dimension in Ω area.

Let us write down the boundary conditions for the one-fourth of a circular ring:

$$\alpha=0,\ \alpha=\frac{\pi}{2}:\quad \mathrm{v}=0,\ A_r=0 \Leftrightarrow \mathrm{v}=0,\ u_{,\alpha}=0, \qquad (3)$$

$$r = r_k,\ \ k = 1,2:\ \ a)\ R_r = F_{11}^{(k)}(\alpha),\ \ R_\alpha = F_{12}^{(k)}(\alpha)\ \text{ or} \\ b)\ u = F_{21}^{(k)}(\alpha),\ \ \text{v} = F_{22}^{(k)}(\alpha),\ \text{or} \\ c)\ \text{v} = F_{22}^{(k)}(\alpha),\ R_r = F_{11}^{(k)}(\alpha),\ \text{or} \\ d)\ u = F_{21}^{(k)}(\alpha),\ R_\alpha = F_{12}^{(k)}(\alpha). \tag{4}$$

It is implied that $F_{11}^{(k)}(\alpha)$, $F_{12}^{(k)}(\alpha)$ functions, together with their first order derivatives, and $F_{21}^{(k)}(\alpha)$, $F_{22}^{(k)}(\alpha)$, functions, together with their first and second order derivatives, decompose into absolute and uniformly convergent Fourier trigonometric series.

Let us consider one of the posed problems, e.g. problem (1), (3), (4a). Other problems can be solved similarly.

3. Solving problem (1), (3), (4a)

Let us consider a plane deformed homogenous isotropic cylinder with r_1 internal radius and r_2 external radius. For generality, let us assume that the cylinder is loaded simultaneously with internal $p_1(\alpha)$ and external $p_2(\alpha)$ stresses. Let us imply that the cylinder is quite long for stress Z_z to be distributed equally across the cross section and the influence of the cylinder bed on the radial relocation is very small. Besides, we consider the case when $Z_z = 0$. Mathematically, this problem is described with formulae (1), (2), (3), (4a). So, we must find the solution of the system of equilibrium equations (1) in $\Omega = \left\{ r_1 < r < r_2, 0 < \alpha < \frac{\pi}{2} \right\}$ area, which satisfies (3), (4a) boundary conditions. We can give the numerical realization of the given problem based on the solutions obtained by using two methods (analytical solution obtained with a method of separation of variables and Lamé solution).

The general solution of system (1) for the considered class of boundary-value problems of elasticity is presented by means of two φ_1 and φ_2 harmonious functions as follows:

$$u = r^3 \varphi_{1,rr} + \frac{\lambda - \mu}{\lambda + \mu} r^2 \varphi_{1,r} + \varphi_{2,r}, \\ \text{v} = r^2 \varphi_{1,r\alpha} + \frac{2(\lambda + 2\mu)}{\lambda + \mu} r \varphi_{1,\alpha} + \frac{1}{r} \varphi_{2,\alpha}, \tag{6}$$

where

$$\varphi_i = \sum_{k=1}^{\infty} \left[\left(A_{ik} r^k + B_{ik} r^{-k} \right) \frac{\sin(k\alpha)}{\cos(k\alpha)} + \left(\tilde{A}_{ik} e^{\alpha k} + \tilde{B}_{ik} e^{-k\alpha} \right) \frac{\sin(k \ln r)}{\cos(k \ln r)} \right] \\ + a_{i0} \alpha \ln r + b_{i0} \ln r + c_{i0} \alpha + d_{i0},\ \ i = 1,2.$$

The stress tensor components will be written down as follows:

$$R_r = (\lambda + 2\mu) r^3 \varphi_{1,rrr} + \lambda r \varphi_{1,r\alpha\alpha} + \frac{5\lambda^2 + 11\lambda\mu + 4\mu^2}{\lambda + \mu} r^2 \varphi_{1,rr} \\ + \lambda \frac{2(\lambda + 2\mu)}{\lambda + \mu} \varphi_{1,\alpha\alpha} + \frac{3\lambda^2 + \lambda\mu - 4\mu^2}{\lambda + \mu} r \varphi_{1,r} + 2\mu \varphi_{2,rr}, \\ A_\alpha = \lambda r^3 \varphi_{1,rrr} + (\lambda + 2\mu) r \varphi_{1,r\alpha\alpha} + \frac{5\lambda^2 + 5\lambda\mu + 2\mu^2}{\lambda + \mu} r^2 \varphi_{1,rr} + \frac{2(\lambda + 2\mu)^2}{\lambda + \mu} \varphi_{1,\alpha\alpha} \\ + \frac{(\lambda - \mu)(3\lambda + 2\mu)}{\lambda + \mu} r \varphi_{1,r} + \frac{2\mu}{r^2} \varphi_{2,\alpha\alpha} + \frac{2\mu}{r} \varphi_{2,r}, \\ R_\alpha = 2\mu r^2 \varphi_{1,rr\alpha} + 4\mu r \varphi_{1,r\alpha} + \frac{2\mu}{r} \varphi_{2,r\alpha} - \frac{2\mu}{r^2} \varphi_{2,\alpha}, \tag{11}$$

The problem to determine displacements and stresses in a thick-walled cylinder is known as Lamé problem, which gained the solution of this problem. [21- 23]

$$u = \frac{1-\mu}{E} \frac{p_1 r_1^2 - p_2 r_2^2}{r_2^2 - r_1^2} r + \frac{1+\mu}{E} \frac{r_1^2 r_2^2}{r} \frac{p_1 - p_2}{r_2^2 - r_1^2}.$$

$$R_r = \frac{p_1 r_1^2 - p_2 r_2^2}{r_2^2 - r_1^2} - \frac{r_1^2 r_2^2}{r^2} \frac{p_1 - p_2}{r_2^2 - r_1^2},$$

$$A_\alpha = \frac{p_1 r_1^2 - p_2 r_2^2}{r_2^2 - r_1^2} + \frac{r_1^2 r_2^2}{r^2} \frac{p_1 - p_2}{r_2^2 - r_1^2},$$

4. Numerical results and consideration

The present paragraph gives the results of the calculation of a thick-walled circular homogenous isotropic body under the impact of internal pressure or external forces on strength. The strength calculation of a cylinder implies determining the minimum thicknesses of the walls of the thick-walled pipes of different materials and different diameters when the stresses in the body do not exceed the admissible value [22, 23], i.e. when the pipe will not disintegrate (is not cracked). So, the numerical results of the boundary-value problems given here and relevant graphs for the one-fourth circular ring when 1) constant normal load is given on the internal boundary and the external boundary is free from loads, 2) constant normal load is given on the external boundary and internal boundary is free from loads. Thus, the dependence of the strength of a plane deformed thick-walled circular cylinder on the thickness and material of the wall (elastic properties) is studied. Its relevant numerical solutions are obtained based on a) the analytical solution obtained with the method of separation of variables, b) Lamé solution.

As it is known, a cylinder is called a thick-walled one, for which the ratio of its wall with its internal diameter is no less that 1/20. Therefore, we must take $0 < r_1 \le \frac{10}{11} r_2$. We will have the variation of the wall thickness, if we fix r_2 and confer r_1 values from range $0 < r_1 \le \frac{10}{11} r_2$. The numerical values are obtained for the following data: $r_2 = 1.5cm,\ 2.5cm,\ 5cm$, $p = -100 kg/cm^2$, and $E = 2 \cdot 10^6 kg/cm^2$, $\nu = 0.3$ for steel, $E = 1.1 \cdot 10^6 kg/cm^2$, $\nu = 0.32$ for copper, $E = 0.7 \cdot 10^6 kg/cm^2$, $\nu = 0.34$ for aluminum and $E = 0.7 \cdot 10^6 kg/cm^2$, $\nu = 0.25$ for grey cast iron.

We had obtained the minimum wall thickness of a circular steel, copper, aluminum and grey cast iron rings (i.e. of a circular cylinder wall), at which the stresses produced in the body do not exceed admissible values, when $r_2 = 1.5,\ 2.5,\ 5$. The obtained numerical values are presented in table 1 for internal load and in table 2 for external load of cylinder.

Table 1: *Minimum admissible thicknesses of a cylinder wall at which the stresses produced in the cylinders of different materials and diameters do not exceed admissible values (internal load)*

Material	Minimum admissible wall thickness ($r_2 - r_1$) cm			E, Kg/cm^2	ν	Admissible stress, Kg/cm^2
	$r_2 = 1.5$	$r_2 = 2.5$	$r_2 = 5$			
Steel	0.1813636	0.3022727	0.6545455	$2 \cdot 10^6$	$\nu = 0.3$	713.801
Copper	0.2263636	0.3772727	0.7545455	$1.1 \cdot 10^6$	$\nu = 0.32$	611.830
Aluminum	0.2713636	0.4522727	0.8545455	$0.7 \cdot 10^6$	$\nu = 0.34$	509.858
Grey cast iron	0.2713636	0.4522727	0.8545455	$0.7 \cdot 10^6$	$\nu = 0.25$	509.858

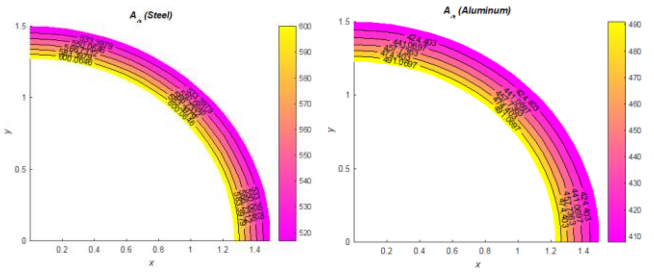
Table 2: *Minimum admissible thickness of a cylinder wall at which the stresses produced in the cylinders of different materials and diameters do not exceed admissible values (external load)*

Material	Minimum admissible wall thickness ($r_2 - r_1$) cm			E, Kg/cm^2	ν	Admissible stress, Kg/cm^2
	$r_2 = 1.5$	$r_2 = 2.5$	$r_2 = 5$			
Steel	0.2263636	0.3772727	0. 7545455	$2 \cdot 10^6$	$\nu = 0.3$	713.801
Copper	0.2713636	0.4522727	0. 8545455	$1.1 \cdot 10^6$	$\nu = 0.32$	611.830
Aluminum	0.3163636	0.5272727	1.054545	$0.7 \cdot 10^6$	$\nu = 0.34$	509.858
Grey cast iron	0.3163636	0.5272727	1.054545	$0.7 \cdot 10^6$	$\nu = 0.25$	509.858

As it was expected and as the tables show, the minimum admissible wall thickness of a steel pipe (cylinder) is less than that of a copper pipe, while the minimum admissible wall thickness of a copper pipe is even less than that of an aluminum pipe. As the Young modulus values for aluminum and grey cast iron are the same, the minimum admissible thicknesses of their walls are also the same. It should also be noted that the greater the pipe diameter is, the more the minimum admissible wall thickness is.

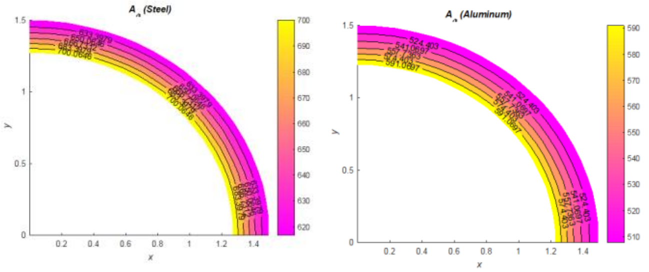
The tables show the minimum thicknesses of a ring (cylinder wall), when the stresses produced in the pipe of different materials (steel, copper, aluminum and grey cast iron) and different diameters (3 cm, 5 cm and 10 cm) do not exceed the relevant admissible values, when 1) constant normal stress is applied to the internal boundary of a ring and the external boundary is free from stresses (Table 1), or 2) constant normal stress is applied to the external boundary and the internal boundary is free from stress (Table 2). These tables show that the minimum admissible pipe wall thickness with the loaded internal boundary is less than with the loaded external boundary.

Figures 1,3,5 present shearing stress isolines for the one-fourth of steel and aluminum circular rings with the diameters of 3 cm, 5 cm and 10 cm. In addition, the internal boundary of the ring is loaded with constant normal force and the external boundary is free from load (internal load). And at Figures 2,4,6 presents shearing stress isolines when constant normal stress is applied to the external boundary of the ring and the internal boundary is free from loads (external load).



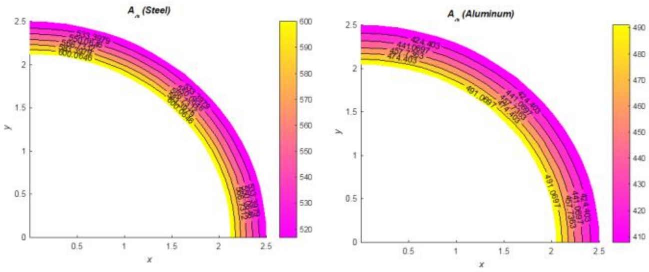
a) b)

Fig. 1 *Shearing stress isolines in one-fourth of a) the steel ring and b) aluminum ring, when $r_2 = 1{,}5$ (internal load).*



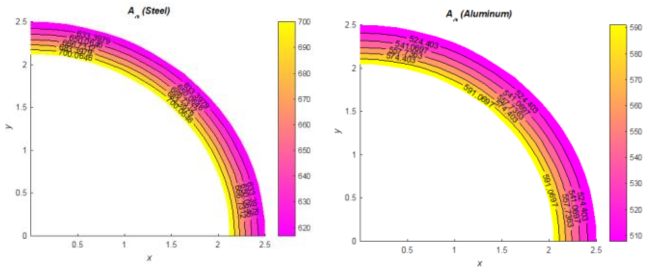
a) b)

Fig. 2 *Shearing stress isolines in one-fourth of a) the steel ring and b) aluminum ring, when $r_2 = 1{,}5$ (external load).*



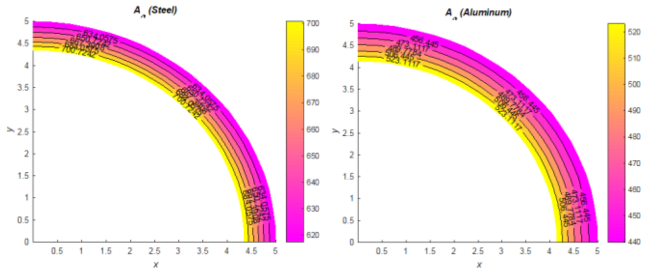
a) b)

Fig. 3 *Shearing stress isolines in one-fourth of a) the steel ring and b) aluminum ring, when $r_2 = 2{,}5$ (internal load).*



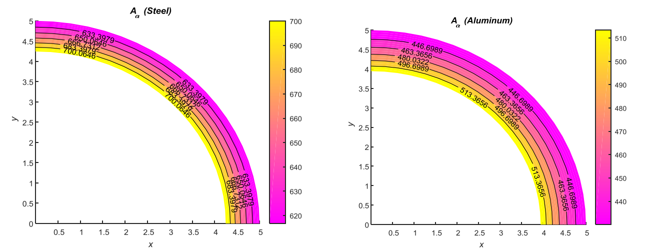
a) b)

Fig. 4 *Shearing stress isolines in one-fourth of a) the steel ring and b) aluminum ring, when $r_2 = 2{,}5$ (external load).*



a) b)

Fig. 5 *Shearing stress isolines in one-fourth of a) the steel ring and b) aluminum ring, when $r_2 = 5$ (internal load).*



a) b)

Fig. 6 *Shearing stress isolines in one-fourth of a) the steel ring and b) aluminum ring, when $r_2 = 5$ (external load).*

5. Conclusion

The principle results presented in the present work can be formulated as follows:

1. The strength of quite a long homogenous isotropic thick-walled circular pipe (cylinder) under the impact of external forces by using the problems of statics of the theory of elasticity is studied. Plane deformation is considered.

2. Mathematical and numerical modeling is provided to study the strength of a thick-walled circular pipe under the impact of external forces by using the problems of statics of the theory of elasticity.

3. The problems of statics of the theory of elasticity are set and solved in the polar coordinate system.

4. An analytical solution is obtained by using the method of separation of variables, which is presented as two harmonious functions.

5. The numerical values are presented and considered and tangential stress isolines are drafted in a one-fourth circular ring, which are obtained by using MATLAB software.

6. The minimum wall thicknesses of the pipes of different materials (steel, copper, aluminum and grey cast iron namely) and diameters (3 cm, 5 cm and 10 cm namely), when the values of stresses originated in them do not exceed the admissible values, are identified.

A cylindrical vessel under the impact of stress is often used as a component in such branches of industry as chemical, military and oil industries and water and nuclear power plants. These components are often subject to a complex load, such as twisting, pressure, temperature, etc. The circular cylinders (pipes) are also widely used in building, machine building, etc. Therefore, the study of the deflected mode of the cylindrical bodies is topical and so, in my opinion, setting the problems considered in the present work and the method of their solution is interesting in a practical view.

References

[1] Bagheri, A., D. Taghizadeh, H. Darijani. On the behavior of rotating thick-walleded cylinders made of hyperelastic materials – Meccanica, 51(3), 2016, 973-692

[2] CHEN Changfu, XIAO Shujun, YANG Yu. Unified solution of limit loads of thick wall cylinder subject to external pressure considering strain softening - Front. Archit. Civ. Eng. China, 1(3), 2007, 329–333

[3] Rakesh Reghunath, Sammon Korah. Analysis of Internally Pressurized Thick Walled Cylinders - Journal of Basic and Applied Engineering Research, 1(2), 2014, 88-93

[4] Sanjeev Sharma, Sanehlata Yadav, Richa Sharma. Creep torsion in thick-walleded circular cylinder under internal and external pressure - Structural Integrity and Life, 18(2), 2018, 89-97

[5] Muskhelishvili, N.I. Some basic problems of the mathematical theory of elasticity, Groningen, Noordhoff, 1953

[6] Khomasuridze, N. Thermoelastic equilibrium of bodies in generalized cylindrical coordinates - Georgian Math. J. 5, 1998, 521–544

[7] Khomasuridze, N, and Zirakashvili, N. Strain control of cracked elastic bodies by means of boundary condition variation. In: Proceedings of International Conference ‘‘Architecture and Construction – Contemporary Problems’’, 30 September–3 October 2010, Yerevan, Jermuk, 158–163 (in Russian)

[8] Zirakashvili, N. Application of the boundary element method to the solution of the problem of distribution of stresses in an elastic body with a circular hole whose interior surface contains radial cracks – Proc. A. Razmadze Math. Inst. 141, 2006, 139–147

[9] Tang, R, and Wang, Y. On the problem of crack system with an elliptic hole - Acta Mech. Sin. 2(1), 1986, 47–53

[10] Zirakashvili, N. Some boundary value problems of elasticity for semi-ellipses – Proc. I. Vekua Inst. Appl. Math. 52,2002, 49–55

[11] Shestopalov, Y. and Kotik, N. Approximate decomposition for the solution of boundary value problems for elliptic systems arising in mathematical models of layered structures. In: *Progress in Electromagnetic Research Symposium*, Cambridge, MA, 26–29 March 2006, 514–518.

[12] Zirakashvili, N. The numerical solution of boundary value problems for an elastic body with an elliptic hole and linear cracks – J. Eng. Math. 65(2), 2009, 111–123

[13] Zirakashvili, N. Study of deflected mode of ellipse and ellipse weakened with crack - ZAMM 97(8), 2017, 932-945. DOI: 10.1002/zamm.201600124

[14] Zirakashvili, N. Analytical solutions of some internal boundary value problems of elasticity for domains with hyperbolic boundaries - Mathematics and Mechanics of Solids 2018, 1-23. DOI: 10.1177/1081286518805269

[15] Zirakashvili N. Analytical solution of interior boundary value problems of elasticity for the domain bounded by the parabola - Bulletin of TICMI, 20 (1), 2016, 3-24

[16] Zirakashvili N. Exact solution of some exterior boundary value problems of elasticity in parabolic coordinates - Mathematics and Mechanics of Solids, 23(6), 2018, 929–943

[17] Jeffery, G.B. Planes tress and planes train in bipolar coordinates – Trans. R. Soc. Lon. 221, 1921, 265–293

[18] Ufland, Ya. S. Bipolar Coordinates in Elasticity. Gostehteoretizdat, Moscow-Leningrad, 1950 (in Russian).

[19] Khomasuridze, N. Solution of some elasticity boundary value problems in bipolar coordinates - Acta Mech. 189, 2007, 207–224

[20] Khomasuridze, N. Representation of solutions of some boundary value problems of elasticity by a sum of the solutions of other boundary value problems - Georgian Mathematical Journal, 10(2), 2005, 257-270

[21] Nowacki, W, Elasicity Theory. Mir, Moscow, 1975 (in Russian)

[22] Feodoseev, V. I. Strength of materials. Nauka, Moscow, 1972 (in Russian)

[23] Shpiro, G. S., Darkov A. V. Strength of materials**.** Vishaia shkola, Moskow, 1989 (in Russian)