EXPONENTIAL STABILITY AND EXACT SOLUTIONS OF THE BOLTZMANN EQUATION IN TWO SPECIAL CASES

ЭКСПОНЕНЦИАЛЬНАЯ УСТОЙЧИВОСТЬ И ТОЧНЫЕ РЕШЕНИЯ УРАВНЕНИЯ БОЛЬЦМАНА В ДВУХ ЧАСТНЫХ СЛУЧАЯХ

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Abstract: *In the first part of the paper it is shown, that in the case of specific initial conditions, solutions of the Cauchy problem for linearized Boltzmann equation have exponential damping when time tends to infinity. In the second part of the paper exact analytic solution of spatially homogeneous linearized Boltzmann equation is built by the use of discrete Laplace transform. The result may be useful in research tasks inside the area of applied rarefied gas dynamics.*

***KEYWORDS**: LINEARIZED BOLTZMANN EQUATION, CAUCHY PROBLEM, EXACT ANALYTIC SOLUTION, EXPONENTIAL DAMPING IN TIME*

1. Introduction

The Boltzmann equation is well known as the basic equation of the kinetic theory of rarefied gases. Already L. Boltzmann understood the role of this equation for substantiating macroscopic models of aerodynamics. In 1916 - 1917 D. Enskog and S. Chapman independently proposed a formal mathematical derivation of macroscopic aerodynamic equations from the Boltzmann equation, including the natural derivation of macroscopic relations for viscosity, thermal conductivity, and diffusion coefficients (the so-called Chapman-Enskog method, which is now considered classical).

On the other hand, the study of the Boltzmann equation as a mathematical object (the theory of the existence and uniqueness of solutions, the analysis of the qualitative properties of solutions, the correctness of the formulation of various problems for this equation, etc.) was noticeably developed only in the second half of the 20th century. The reason for this was, in our opinion, mainly two circumstances.

First, the real need for priority use of the kinetic approach when considering practical problems of aerodynamics arose with the development of rocket and space technology and high-altitude aircraft, vacuum technology, etc. With a sufficiently high degree of gas dilution (when the average free path of a molecule becomes comparable to the size of a body moving in a gas, which for the earth’s atmosphere corresponds to altitudes greater than 100 km.), Classical aerodynamics should be replaced by the kinetic theory of gases. The basis of this theory, as already noted, is the Boltzmann equation. Thus, when solving practical engineering problems related to the motion of bodies in the high-altitude layers of the atmosphere, it became necessary to develop adequate approximate, in particular, numerical methods for solving the Boltzmann equation. But for the reliable use of certain approximate methods, their rigorous justification is necessary, which cannot be obtained without mathematically accurate results on the existence and properties of solutions of this equation.

Secondly, the mathematical apparatus, which proved to be suitable for a rigorous analysis of the Boltzmann equation, was sufficiently developed only in the second half of the last century. First of all, we are talking about the methods of functional analysis, related, in particular, to the theory of unbounded operators in Banach spaces, the theory of one-parameter semigroups of operators, the theory of generalized functions. At the same time, many mathematical problems associated with the Boltzmann equation, are not too tough for even advanced methods of functional analysis.

It can be stated that, despite the serious successes achieved by the works of not too numerous foreign and domestic researchers, the mathematical theory of the Boltzmann equation is still far from any kind of completed. From this point of view, even partial results relating to the mathematical theory of the Boltzmann equation can be very informative and practically useful.

This paper is devoted to two mathematical problems related to the Boltzmann equation. The first concerns the asymptotic properties of solutions of the linearized Boltzmann equation on an infinite time interval, the second – to the algorithm for constructing an exact solution of the spatially homogeneous Boltzmann equation.

2. The exponential stability of solutions of the Cauchy problem for the linearized Boltzmann equation

Consider the Cauchy problem for the linearized Boltzmann equation of the kinetic theory of gases [1, 2]:

(1) $$\frac{\partial f}{\partial t}+\vec{u}\cdot\frac{\partial f}{\partial \vec{x}}=L\left[f\right],\quad t>0,\quad L\left[f\right]=\mathrm{K}\left[f\right]-\nu f\ ,$$
$$f=f(\vec{x},\vec{u},t),\quad \vec{x}\in\mathbf{R^3},\quad \vec{u}\in\mathbf{R^3},\quad t\geq 0\,,$$

(2) $$f\big|_{t=0}=f_0(\vec{x},\vec{u})$$

Here $f(\vec{x},\vec{u},t)$ - linearized distribution of molecules by coordinates $\vec{x}$ and velocities $\vec{u}$ at the moment of time t. $\mathrm{K}\left[f\right]$- linear bounded operator acting on f as a function of $\vec{u}$; $\nu=\nu(u)=O(u^{\beta})$ at $u\to\infty$, $0<\beta\leq 1$, $u=\left|\vec{u}\right|$. The properties of the function $\nu(u)$ depend on the specific model of intermolecular interaction, taken in the derivation of the kinetic equations. See details in [1, 2].

It is known [3a, b] that the solution to problem (1), (2) is - in the case of "hard" intermolecular interaction potentials $U:\ r^{-k}, k>4$ - has at $t\to\infty$ in general, the power asymptotic of the form $O\left(\dfrac{1}{1+t^{\mu}}\right), \mu>0$. This result is obtained under the assumption that $f(\vec{x},\vec{u},t)$ at $x=\left|\vec{x}\right|\to\infty$ behaves like a function from $L_p(\mathbf{R}_x^{\mathbf{3}}),\ p>1$.

It turns out that if we impose more stringent requirements on behavior of $f(\vec{x},\vec{u},t)$ at $x\to\infty$, for example, require that $f(\vec{x},\vec{u},t)$ satisfies by $\vec{x}$ (and evenly by $\vec{u},\ t$) a condition of type

(3) $$f(\vec{x},\vec{u},t)=O\left(\exp\left(-\alpha\left|\vec{x}\right|^{1+\varepsilon}\right)\right),\quad \left|\vec{x}\right|\to\infty\,,$$
$$\alpha>0,\quad \varepsilon>0\,,$$

i.e. equilibrium ($f \to 0$ at $t \to \infty$) is setting exponentially fast.

The idea of the proof is as follows (we use notation from [7, 8]). We shell seek $f(\vec{x},\vec{u},t)$ in a class of functions such that for almost all $\vec{u} \in \mathbf{R^3}$ and all $t > 0$ $f(\vec{x},\vec{u},t) \in \boldsymbol{E}\,'_x$, i.e. the function f can be represented as (see [7, 8])

(4) $$f(\vec{x},\vec{u},t) = \sum_{r=0}^{\infty}\sum_{|q|=r} c^{(q)}(\vec{u},t)\,\delta^{(q)}(\vec{x}) .$$

Substituting this expression into (1) and taking into account Theorem 3.1 and formula (3.3) from [8], we get infinite "hooking" system of equations for coefficients $c^{(q)}(\vec{u},t)$:

$(5)_1$ $$\frac{\partial c^{(0)}}{\partial t} = L[c^{(0)}] ,$$

$(5)_2$ $$\frac{\partial c^{(q)}}{\partial t} = L[c^{(q)}] - [u_1 c^{(q-I_1)} + u_2 c^{(q-I_2)} + u_3 c^{(q-I_3)}], \quad |q| \neq 0$$

where I_1, I_2, I_3 denote multi-indices (1, 0, 0), (0, 1, 0) and (0, 0, 1) respectively.

Equations $(5)_2$ represent inhomogeneous equations of the form

$$\frac{\partial c^{(q)}}{\partial t} = L[c^{(q)}] - g_q(\vec{u},t), \quad |q| \neq 0 ,$$

where $g_q(\vec{u},t)$ - known function (at each step - its own). Thus, the properties of functions $c^{(q)}(\vec{u},t)$ depend on the properties of the operator L. The latter have been thoroughly studied [1-6]. In particular, the operator L on the subspace of functions $w(\vec{u},t)$, orthogonal in the sense of $L_2(\mathbf{R}_u^3)$ subspace of additive invariants (which is essentially equivalent to the implementation of the classical conservation laws for the mass of gas), generates a semigroup $T(t), t > 0$ of bounded operators [9], solving an abstract Cauchy problem for an equation $(5)_1$; it turns out $\|T(t)\| \leq const \cdot e^{-\mu t}, \mu > 0$. By a method similar to that used in [3, 4], by induction, we obtain, for solutions of equations $(5)_2$ an estimate of the form (the norm is understood in the sense of $L_2(\mathbf{R}_u^3)$): $\|c^{(q)}(t)\| \leq const \cdot e^{-\gamma t}, \gamma > 0$, where *const* depends on the initial distribution function $f_0(\vec{x},\vec{u})$ and parameters of the operator *L*. The latter estimate, taking into account Theorem 1.8 from [7], allows us to make a conclusion about exponentially fast (in time) establishing equilibrium in the system described by task (1) - (2).

3. The exact analytical solution of the spatially homogeneous linearized Boltzmann equation

Consider the Cauchy problem for a spatially homogeneous linearized Boltzmann equation in the case of “hard” intermolecular interaction potentials $U: r^{-k}, k > 4$ (hereinafter, notation and terminology see, for example, in [1, 2]):

(6) $$\frac{\partial f}{\partial t} = L[f], \quad t > 0, \quad L[f] = \mathrm{K}[f] - \nu f ,$$

$$f = f(\vec{u},t), \quad \vec{u} \in \mathbf{R^3}, \quad t > 0 ,$$

(7) $$f(\vec{u},0) = f_0(\vec{u})$$

Here $f(\vec{u},t)$ - is the linearized velocity distribution function of the molecules at the moment of time t. $\mathrm{K}[f]$- linear bounded operator acting on f as a function of $\vec{u}$; $\nu = \nu(u) = O(u^{\beta})$ when $u \to \infty$, $0 < \beta \leq 1$, $u = |\vec{u}|$. Properties of the function $\nu(u)$ depend on the specific model of intermolecular interaction, taken in the process of derivation of the kinetic equations. See details in [1, 2]. The requirements of the "rigidity" of potentials determine the properties of the operator $\mathrm{K}[f]$ and the function $\nu(u)$. This problem has been studied quite well from a mathematical point of view with respect to questions of correctness and qualitative properties of solutions to the problem (6)-(7) (see for example, [1, 2, 10, 11]).

However, there remains the question important not only from a theoretical, but also from a practical point of view: the question of constructing (at least in special cases) exact analytical solutions of this equation. This is due, in particular, to the fact that the general theorems of existence and uniqueness for the Boltzmann equation, due to its complexity, give results that are not well adapted to the problems of *constructing* its solutions (even approximate). A sufficiently complete overview of these results can be found in [12–14] (little has changed regarding the questions of interest to us since these reviews were written). Numerical methods for solving the Boltzmann equation are based, as a rule, on statistical modeling methods, and are far from always strictly justified [15, 16]. On the other hand, the presence in the stock of exact analytical solutions of even particular problems can be useful for the analysis of mathematical models of more general processes. Here we will propose a solution to a similar problem - the construction of an exact analytical solution of the problem (6) – (7).

We introduce the notation

$$\varphi_k = \varphi_k(\vec{u}) = \left.\frac{\partial^k f}{\partial t^k}\right|_{t=0}, \quad k = 0,1,2,..., \quad \varphi_0(\vec{u}) = f_0(\vec{u}) .$$

We will seek a solution of (6) – (7) in the form

(8) $$f(\vec{u},t) = \sum_{k=0}^{\infty} (\varphi_k / k!) t^k .$$

It is clear that if series (8) converges absolutely and uniformly (with respect to $\vec{u} \in \mathbf{R^3}$), then we obtain the solution of problem (6) - (7) that is analytic in *t*. Let us prove the convergence of the series (8). Let us turn to φ_k in (6) and then to the discrete Laplace transform [17, ch. 8]:

$$\varphi_{k+1} = K[\varphi_k] - \nu\varphi_k ;$$

$$\Phi^*(q,\vec{u}) = \sum_{k=0}^{\infty} e^{-qk}\varphi_k(\vec{u}), \quad \mathrm{Re}\, q > 0 .$$

For $\Phi^*(q,\vec{u})$ we obtain, therefore, the equation

$$\Phi^* = \frac{1}{\nu + e^q}\left(K[\Phi^*]\right) + \varphi_0 \equiv A_q[\Phi^*].$$

It is easy to understand that with sufficiently large $\mathrm{Re}\, q$ the operator A_q will be contractive (in any $\vec{u}$ - space, in which the operator K is bounded). Hence follows the boundedness of the function Φ^* by *q*. But then there exists an inverse transformation

$$\varphi_k = \frac{1}{2\pi i}\int_{s-i\pi}^{s+i\pi} \Phi^*(q,\vec{u}) e^{qk} dq ,$$

from which we get that

$$|\varphi_k(\vec{u})| \leq \rho e^{\mu k} ,$$

where ρ and μ – are constants. From here we immediately obtain the absolute and uniform - by $\vec{u}$ - convergence of the series (8), as required. Not without interest is, in our opinion, the fact of analyticity of the solution with respect to time.

4. Literature

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