MATHEMATICAL AND NUMERICAL SIMULATION OF STRESSES AND DISPLACEMENTS LOCALIZATION PROBLEMS

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Abstract: *Mathematical and numerical simulation of the non-classical problems, namely problems of localization of stresses and displacements in the elastic body, are obtained by the boundary element method. The current work examines two localization problems, which have the following physical sense: on the middle point of the segment lying inside a body parallel to the border half plane in first case a point force is applied, and we must find such value of the normal stress along the section of the border half plane, which will cause this point force, while in the second case, there is given a vertical narrow deep trench outgoing of this point, and we must find such value of the normal stress along the section of the border half plane, which will result in such a pit. By using MATLAB software, the numerical results are obtained and corresponding graphs are constructed.*

Keywords: NON-CLASSICAL PROBLEM, BOUNDARY ELEMENT METHOD, LOCALIZATION PROBLEM, HOMOGENEOUS ISOTROPIC HALF PLANE

1. Introduction

In the present work, mathematical and numerical simulation of the problems of localization of stresses and displacements in a body, are gained by the boundary element method (BEM) [1]. In a certain sense, the problem of localization of stresses in the elastic body is the inverse problem to the delocalization problem [2]. The localization problem is defined as follows: to change a sufficiently uniform stressed-deformed state of a body for a sharply expressed non-uniform stressed-deformed state (in conditions of constant external perturbations) by changing and appropriate selection of parameters of the medium.

In the theory of elasticity, there are a number of problems [3]-[10] that could be called non-classical due to the fact that boundary conditions on a part of the boundary surface or on the entire boundary surface are either over-determined or underdetermined, or the conditions on the boundary are connected with the conditions inside the body (so called non-local problems).

The current article sets and solves non-classical two-dimensional elasticity problems by using BEM, and problems of localization of stress and displacement for a homogeneous isotropic elastic half-plane are formulated based on them. The present paper examines two localization problems, which have the following physical meaning: on the middle point of the segment lying inside a body parallel to the border half plane in first case a point force is applied, and we must find such value of the normal stress along the section of the border half plane, which will cause this point force (stresses localization), while in the second case, there is given a vertical narrow deep trench outgoing of this point, and we must find such value of the normal stress along the section of the border half plane, which will result in such a pit (displacements localization).

Finally, there are test examples given showing the value of normal stress supposed to apply to the section of the half-plane boundary to obtain the pre-given localized stress or displacement at the midpoint of the segment inside the body. The numerical results of these problems are obtained and presented appropriate graphs, and mechanical and physical interpretations of the problems.

2. Formulation of problems

Let us set some non-classical static problems for homogeneous isotropic half plane (see. Fig.1).

It is known that a homogeneous system of elastic static equilibrium in displacements in the Cartesian system of coordinates has the form [16]

$$\begin{cases}(\lambda+\mu)\theta_{,x}+\mu\Delta u=0\\ (\lambda+\mu)\theta_{,y}+\mu\Delta \mathrm{v}=0\end{cases} \quad \text{in } D \qquad (1)$$

where $\lambda=\dfrac{\nu E}{(1-2\nu)(1+\nu)}$, $\mu=\dfrac{E}{2(1+\nu)}$ are Lamé constants, E is elasticity modulus, and ν Poissons's ratio; $\Delta(\cdot)=(\cdot)_{,xx}+(\cdot)_{,yy}$ is a Laplacian, $\theta=\operatorname{div}\vec{U}=u_{,x}+\mathrm{v}_{,y}$; $\vec{U}=(u,\mathrm{v})$ is the displacement vector; $(\cdot)_{,x}=\dfrac{\partial(\cdot)}{\partial x}$, $(\cdot)_{,y}=\dfrac{\partial(\cdot)}{\partial y}$; $(\cdot)_{,xx}=\dfrac{\partial^2(\cdot)}{\partial x^2}$; $(\cdot)_{,yy}=\dfrac{\partial^2(\cdot)}{\partial y^2}$.

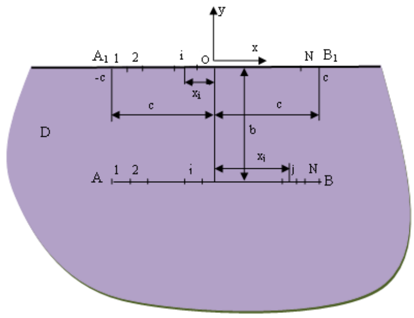


Fig. 1 *Illustration of localization problems of stresses and displacements for elastic half plane.*

2.1. Statement and solving of problem when normal stress is applied to segment inside half plane

(a) Setting. Let us consider a non-classical problem for half plane D (see Fig. 1), when the tangent stress along the entire border and normal stress along boundary segment $|x|>c,\ y=0$ equal to zero. Along segment $|x|\le c,\ y=-b$ inside the body, the value of normal stress σ_{yy} is known. So, let us find the solutions to the system of equilibrium equations (1) satisfying the following boundary conditions:

$$\text{for } |x|<\infty \text{ and } y=0:\quad \sigma_{yx}=0,$$
$$\text{for } |x|>c \text{ and } y=0:\quad \sigma_{yy}=0,$$
$$\text{for } |x|\le c \text{ and } y=-b:\quad \sigma_{yy}=-P_0(x),$$

where $P_0(x)$ is the sufficiently smooth function given along segment $[-c;c]$.

We can formulate the set problem as follows: let us find the kind of distribution of normal stress σ_{yy} along section $|x|\le c,\ y=0$ of the boundary of a half plane (see Fig. 1) so that the normal stress along segment $|x|\le c,\ y=-b$ inside the body equals to the values of given function $P_0(x)$.

If we consider function of kind $P_0(x)=P\cdot 10^{-|4x|}$ $(P=\text{constant})$, which describes a force similar to the concentrated one, then we will have the following localization problem: we must find the kind of distribution of normal stress σ_{yy} along segment A_1B_1 to obtain the concentrated force of the given value (localization of stresses) along section AB (see Fig. 1).

(b) Solution. Let us divide segments $|x| \le c,\ y = 0$ and $|x| \le c,\ y = b$ into N segments (elements) of the same size $2a$ and smaller sizes (i.e. $a = c / N$). We mean that constant normal stresses P_y^j act on each j th element of length $2a$ with center $(x^j;0)$ of segment $|x| \le c,\ y = 0$. We need to find such values of these stresses, for which the values of the normal stresses in middle points $(x^i, -b)$ of each i th segment with a length of $2a$ along segment $|x| \le c,\ y = -b$ inside body will equal to the given value of $-P_0(x^i)$.

Normal stress in the centre of the i th element lying on segment $|x| \le c,\ y = -b$ will equal to following sum:

$$\sigma_{yy}(x^i, -b) = \sum_{j=1}^{N} A^{ij} P_y^j, \quad i = 1, 2, ..., N,$$

where for the influence coefficients A^{ij} has the following formula

$$A^{ij} = -\frac{1}{\pi}\left[\left(\arctan\frac{b}{x^i - x^j - a} - \arctan\frac{b}{x^i - x^j + a}\right)\right.$$
$$\left. + \frac{b(x^i - x^j + a)}{(x^i - x^j + a)^2 + b^2} - \frac{b(x^i - x^j - a)}{(x^i - x^j - a)^2 + b^2}\right].$$

Thus, we obtain the following system of N linear algebraic equations with N unknown quantities P_y^j , $j = 1, 2, \ldots, N$.

$$\sum_{j=1}^{N} A^{ij} P_y^j = P_0(x^i), \quad i = 1, 2, \ldots, N. \tag{2}$$

If solving (2) system in relation to unknown quantities P_y^j by means of any standard method of numerical analysis (by method of Gauss in our case), then we can assume that the set problem is solved and $\sigma_{yy}^j = P_y^j, \quad j = 1, \ldots, N$.

After solving these equations, we can express the displacements and stresses at any point (x^i, y^k) of the body by means of other linear combination of load P_y^j . For example, the stresses and displacements have the following form:

$$\sigma_{xx}(x^i, y^k) = \frac{1}{\pi}\sum_{j=1}^{N}\left[\left(\arctan\frac{y^k}{x^i - x^j + a^j} - \arctan\frac{y^k}{x^i - x^j - a^j}\right)\right.$$
$$\left. - \frac{y^k(x^i - x^j + a^j)}{(x^i - x^j + a^j)^2 + (y^k)^2} + \frac{y^k(x^i - x^j - a^j)}{(x^i - x^j - a^j)^2 + (y^k)^2}\right]P_y^j,$$

$$\sigma_{yy}(x^i, y^k) = \frac{1}{\pi}\sum_{j=1}^{N}\left[\left(\arctan\frac{y^k}{x^i - x^j - a^j} - \arctan\frac{y^k}{x^i - x^j + a^j}\right)\right.$$
$$\left. - \frac{y^k(x^i - x^j + a^j)}{(x^i - x^j + a^j)^2 + (y^k)^2} + \frac{y^k(x^i - x^j - a^j)}{(x^i - x^j - a^j)^2 + (y^k)^2}\right]P_y^j, \tag{3}$$

$$\sigma_{xy}(x^i, y^k) = \frac{1}{\pi}\sum_{j=1}^{N}(y^k)^2\left[\frac{1}{(x^i - x^j + a^j)^2 + (y^k)^2}\right.$$
$$\left. - \frac{1}{(x^i - x^j - a^j)^2 + (y^k)^2}\right]P_y^j,$$
$$i = 1, 2, ..., M_1, \quad k = 1, 2, ..., M_2.$$

$$u_x^j(x^i, y^k) = -\frac{1}{2\pi\mu}\sum_{j=1}^{N}\left\{(1 - 2\nu)\left[(x^i - x^j - a^j)\arctan\frac{y^k}{x^i - x^j - a^j}\right.\right.$$
$$\left. - (x^i - x^j + a)\arctan\frac{y^k}{x^i - x^j + a^j} - \pi a\right]$$
$$\left. + (1 - \nu)y^k \ln\frac{(x^i - x^j - a^j)^2 + (y^k)^2}{(x^i - x^j + a^j)^2 + (y^k)^2}\right\}P_y^j,$$

$$u_y^j(x^i, y^k) = \frac{1}{2\pi\mu}\sum_{j=1}^{N}\left\{-y^k(1 - 2\nu)\left(\arctan\frac{y^k}{x^i - x^j - a^j} - \arctan\frac{y^k}{x^i - x^j + a^j}\right)\right.$$
$$+ (1 - \nu)\left[(x^i - x^j - a^j)\ln\left((x^i - x^j - a^j)^2 + (y^k)^2\right)\right.$$
$$- (x^i - x^j + a^j)\ln\left((x^i - x^j + a^j)^2 + (y^k)^2\right)$$
$$\left.\left. + (L - x^j + a^j)\ln(L - x^j + a^j)^2 - (L - x^j - a^j)\ln(L - x^j - a^j)^2\right]\right\}P_y^j.$$

2.2. Statement and solving of problem when normal displacement is applied to segment inside half plane

(a) Setting. Let us consider a non-classical problem, when along the entire border of half plane D (see Fig. 1) the tangent stress is equal to zero, and normal displacement u_y on segment $|x| \le c, \quad y = -b$ lying inside the body is known. Also, normal stress along part $|x| > c,\ y = 0$ of boundary is equal to zero. Thus, we have the following boundary conditions:

when $|x| < \infty$ and $y = 0$: $\sigma_{yx} = 0,$
when $|x| > c$ and $y = 0$: $\sigma_{yy} = 0,$
when $|x| \le c$ and $y = -b$: $u_y = -U_0(x),$

where $U_0(x)$ is the sufficiently smooth function given along segment $[-c, c]$.

We can formulate this problem as follows: let us find the distribution of normal stress σ_{yy} along part $|x| \le c,\ y = 0$ of the boundary of the half plane when normal displacement along segment $|x| \le c,\ y = -b$ lying inside half plane D equals to $-U_0(x)$.

Let us consider this function of the following kind $U_0(x) = P \cdot 10^{-|4x|}$, $(P = \text{constant})$, which describes clearly expressed non-uniform normal displacement. Thus, we will have the following localization problem: let us find the distribution of normal stress σ_{yy} along segment A_1B_1 to obtain the pit of a given value along segment AB (displacements localization) (see Fig. 1).

(b) Solution. Let us divide segments $|x| \le c,\ y = 0$ and $|x| \le c,\ y = -b$ into N segments (elements) with equal $2a$ and smaller lengths. We mean that constant normal stresses P_y^j act on each j th segment of segment $|x| \le c,\ y = 0$, each with the length of $2a$ and with centre $(x^j, 0)$. We must find such values of these stresses, for which the values of normal displacement in middle point $(x^i, -b)$ of each i th element with length $2a$ of $|x| \le c,\ y = -b$ segment inside the body should equal to the given value of $-U_0(x^i)$.

Normal displacement in the centre of the i th element lying on segment $|x| \le c,\ y = -b$ will be computed with the following formula:

$$u_y(x^i, -b) = \sum_{j=1}^{N} B^{ij} P_y^i, \qquad i = 1, 2, ..., N,$$

where we have the following formula for influence coefficients B^{ij} :

$$B^{ij} = \frac{1}{2\pi\mu}\left\{-b(1 - 2\nu)\left(\arctan\frac{b}{x^i - x^j - a} - \arctan\frac{b}{x^i - x^j + a}\right)\right.$$
$$+ (1 - \nu)\left[(x^i - x^j - a)\ln\left((x^i - x^j - a)^2 + b^2\right)\right.$$
$$- (x^i - x^j + a)\ln\left((x^i - x^j + a)^2 + b^2\right)$$
$$\left.\left. + (L - x^j + a)\ln(L - x^j + a) - (L - x^l - a)\ln(L - x^j - a)^2\right]\right\}.$$

Thus, the set problem is reduced to solving the following system of linear algebraic equations (N equations with N unknown values):

$$\sum_{j=1}^{N} B^{ij} P_y^j = -U_0(x^i), \qquad i = 1, 2, ..., N. \tag{4}$$

If we solve system (4) in relation to unknown values P_y^j, then the set problem can be considered as solved, like the problem set in 2.1.

3. Numerical simulation

By using MATLAB software, we obtained the numerical values of the normal stresses (problem of stresses localization) and displacements (problem of displacements localization) along segment AB (the given normal load and normal displacement) and distribution of normal stresses along segment A_1B_1 (the obtained normal stress) shown in Fig. 1 for the following data: c=1m, 2m, 3m, 4m, 15m, 18m, 20m, 30m, and b=5m, 6,5m, 8m, 10m, 15m 18m, 20m, 30m; $N = 120$; $P = 10\,kg/cm^2$. Below are graphs of some of the obtained results. Namely, Fig. 2 shows load $P_0(x)$ and Fig. 3, Fig. 4 shows normal displacement $U_0(x)$ along AB segment and distribution of obtained normal stress P_y along A_1B_1 segment, when $c = 1m$ and $b = 5m, 6{,}5m, 8m, 10m$.

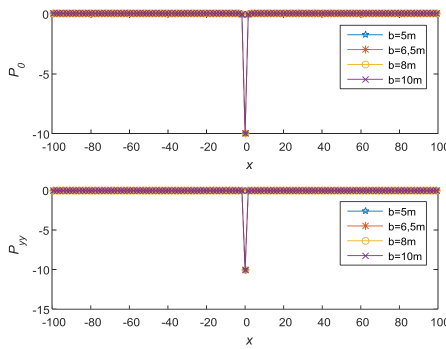


Fig. 2 The load $P_0(x)$ along segment AB and distribution of obtained normal stress $P_{yy} := \sigma_{yy}$ along segment A_1B_1, when $c = 1m$.

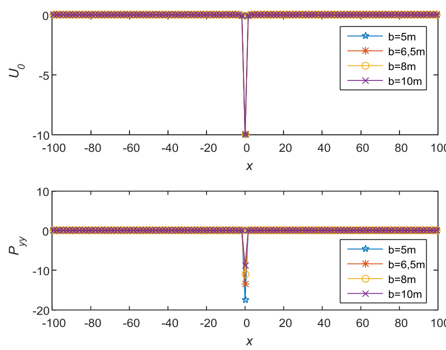


Fig.3 Displacement $U_0(x)$ along segment AB and distribution of obtained normal stress P_{yy} along segment A_1B_1, when $c = 1m$ and $E = 2\times10^2\,kg/cm^2$, $\nu = 0.42$ (technical rubber).

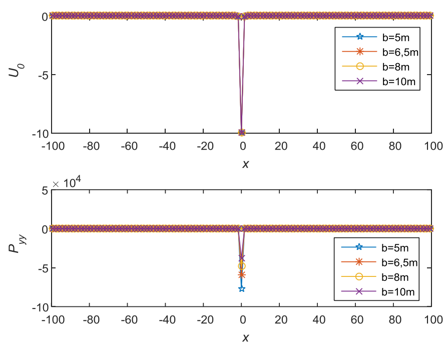


Fig. 4 Displacement $U_0(x)$ along segment AB and distribution of obtained normal stresses P_{yy} along segment A_1B_1, when $c = 1m$ and $E = 2\times10^6\,kg/cm^2$, $\nu = 0.3$ (steel).

Besides, represented 3D graphs of the distribution of stresses and displacements in the body section relevant to domain $-c < x < c$, $-30 < y < -10$, when $c = 1m$, $b = 30m$; for steel $E = 2\times10^6\,kg/cm^2$, $\nu = 0.3$ (see Fig. 5 and Fig. 8 for stresses localization problem, and Fig. 6 and Fig. 10 for displacements localization problem) and technical rubber $E = 2\times10^2\,kg/cm^2$, $\nu = 0.42$ (see Fig. 9 for stresses localization problem, and Fig.7, Fig.11 for displacements localization problem). Formula (3) evidences that the stresses in the stress problems do not depend on Young's modulus and Poison's ratio. As for the displacements, the normal displacement less and tangential displacement is bigger in steel than in technical rubber.

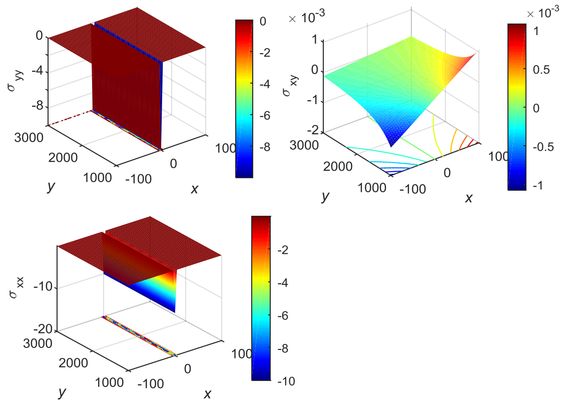


Fig. 5 Distribution of stresses in domain $-c < x < c$, $-30 < y < -10$, when $c = 1m$, $b = 30m$, $\nu = 0.3$ (in stresses for the problem, when $P_0(x) = P\cdot10^{-|4x|}$).

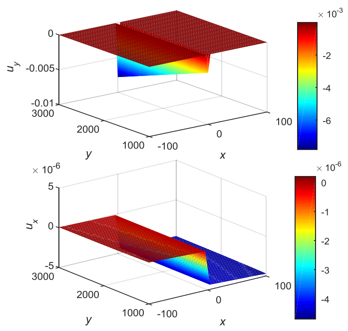


Fig. 6 Distribution of displacements for steel in domain $-c < x < c$, $-30 < y < -10$, when $c = 1m$, $b = 30m$, $E = 2\times10^6\,kg/cm^2$, $\nu = 0.3$ (in stresses for the problem, when $P_0(x) = P\cdot10^{-|4x|}$).

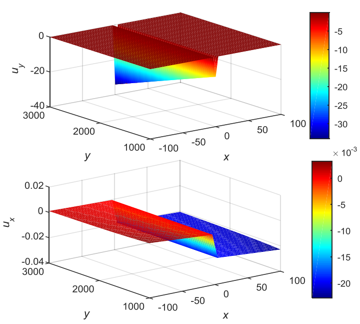


Fig. 7 Distribution of displacements for technical rubber in domain $-c < x < c$, $-30 < y < -10$, when $c = 1m$, $b = 30m$, $E = 2\times10^2\,kg/cm^2$, $\nu = 0.42$ (in stresses for the problem, when

$P_0(x) = P \cdot 10^{-|4x|}$).

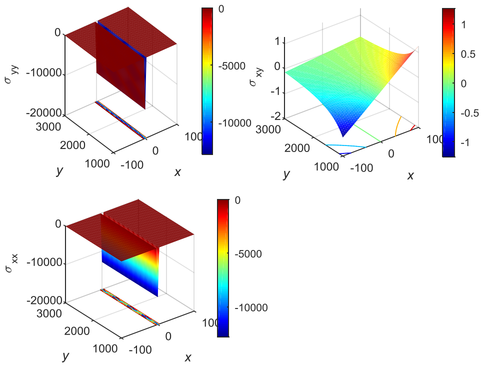
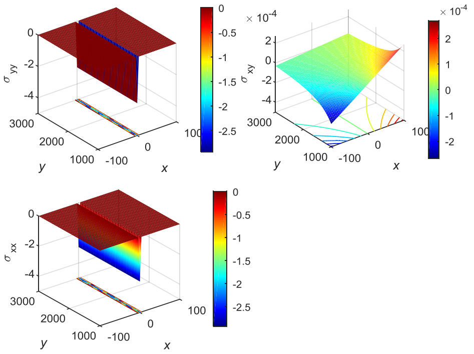


Fig. 8 *Distribution of stresses in the part of the body of steel bordered by domain $-c < x < c$, $-30 < y < -10$, when $c = 1m$, $b = 30m$, $E = 2\times10^6\ kg/cm^2$, $\nu = 0.3$ (in displacements for the problem when $P_0(x) = P \cdot 10^{-|4x|}$).*



Fi. 9 *Distribution of stresses in the part of the body of technical rubber bordered by domain $-c < x < c$, $-30 < y < -10$, when $c = 1m$, $b = 30m$, $E = 2\times10^2\ kg/cm^2$, $\nu = 0.42$ (in displacements for the problem when $P_0(x) = P \cdot 10^{-|4x|}$).*

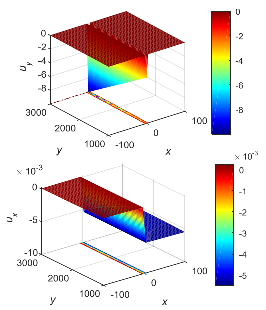


Fig. 10 *Distribution of displacements in the part of the body of steel bordered by domain $-c < x < c$, $-30 < y < -10$, when $c = 1m$, $b = 30m$, $E = 2\times10^6\ kg/cm^2$, $\nu = 0.3$ (in displacements for the problem when $P_0(x) = P \cdot 10^{-|4x|}$).*

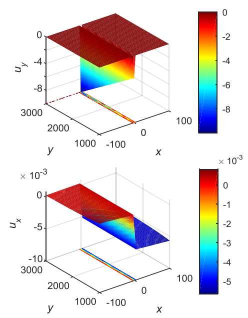


Fig. 11 *Distribution of displacements in the part of the body of technical rubber bordered by domain $-c < x < c$, $-30 < y < -10$, when $c = 1m$, $b = 30m$, $E = 2\times10^2\ kg/cm^2$, $\nu = 0.42$ (in displacements for the problem when $P_0(x) = P \cdot 10^{-|4x|}$).*

4. Conclusion

The paper sets non-classical problems, and problems of localization of stress and displacement for a homogeneous isotropic elastic half-plane are formulated based on them. The essence of the problems is as follows: we must find the distribution of the normal stress along section A_1B_1 (see Fig. 1) of the border of the half plane so that normal stress σ_{yy} or normal displacement u_y along segment AB parallel to the border of a given length distanced from the border by b within the body should equal to the value of the given function. If we take the kind of this function, which describes the point-force applied to the middle point of section AB (e.g. $U_0(x) = C \cdot 10^{-|4x|}$, $(C = \text{constant})$), we will obtain the problem of localization of stresses and displacements. The set problems are solved by BEM [1].

By using the MATLAB's software, we obtained the numerical results and plotted the corresponding graphs showing the values of normal stress to be applied to the part of the boundary of the half plane to obtain the point force or displacement in the middle point of a segment inside the body. The paper also presents 3D graphs of distribution of stresses and displacements within the parts of the bodies of steel and technical rubber bordered by domain .

The problems considered in the work can be used in practice, e.g. in soils and rocks, materials that are susceptible to cracking and faulting when sheared, as well materials used to demolish military structures or in underground facilities.

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