ON THE ASYMPTOTIC IN TIME OF SOLUTIONS OF THE BOLTZMANN EQUATION IN THE CASE OF SOFT INTERMOLECULAR POTENTIALS

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Abstract: *The work is devoted to the mathematical problems of the analysis of asymptotic time behavior of solutions of the nonstationary Boltzmann equation. The proof of the fundamental difference between such behavior for the cases of "hard" and "soft" (in the sense of H. Grad) potentials of intermolecular interaction is given*

KEYWORDS: BOLTZMANN EQUATION, SOFT POTENTIALS, SOLUTIONS, ASYMPTOTIC IN TIME

1. Introduction

Behavior of the solutions of the Boltzmann equation

$$\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}=Q(F,F);\ F|_{t=0}=F_0,\ x\in R_x^3,\ u\in R_u^3 \quad (1)$$

at large values of time are considered in most serious studies of this object. In fact, even Boltzmann expressed his thoughts on the possibility of rapid relaxation of an arbitrary initial distribution function to equilibrium. Many physicists now adhere to this conclusion, although the evidence they use is often far from mathematical perfection. The first serious analysis of these issues was carried out by Carleman [1] as early as the 30s of the XX century, and then only after 30 years was continued by many researchers. A fairly complete review of the results is contained in [2, 3]. In the aspect that interests us, their essence is that for solution *F* of problem (1), an inequality of the form

$$N(F-F_M)\le C_0\,p(t) \qquad (2)$$

is true, where N – suitable norm in the space of functions depending on speed u and radius-vector x (so $N(F)$ – is time dependent function); C_0 – is a constant, depending possibly on the initial distribution $F_0(x,u);F_M=F_M(|u|)$ – Maxwell distribution; the behavior of functions $p(t)$ essentially depends, on the one hand, on the class of function spaces in which a solution is sought, and on the other, on the properties of the collision operator $Q(F,F)$, characterized by assumptions about the type of intermolecular interaction potential.

For "hard" "cut off in the corner" potentials $U\sim r^{-k}$, $k>5$ the problem was investigated very actively; the main result is that the function $p(t)$ in (2) tends to zero with an infinite increase in time *t* either as a power law or as an exponent, depending on the degree of smoothness over the coordinates of the initial distribution, the boundedness (or not) of the spatial region and the rate of decrease $F_0(x,u)$ at $|u|,|x|\to\infty$. The presented results are well known [2, 3, 4, 5] (see also chapter 1 of [6]).

Since in what follows we will consider situations close to equilibrium, then, as usual, instead of a function *F*, we use $f=F_M^{-\frac{1}{2}}(F-F_M)$. Equation (1) goes over to

$$\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}=L(f)+\nu(|u|)\Gamma(f,f);\ f_{t=0}=f(x,u) \quad (3)$$

(see, for example, [2, 3] and chapter 1 of [6]).

The result (2) in terms of a function f has the form

$$N(f)\le N_1(f_0)\,p(t), \qquad (4)$$

where N_1 – a norm that is generally different from the norm N (the properties of a solution *f* generally speaking, worsen compared with the properties of the initial function f_0 – see Caflich's work [7]), and $p(t)\to 0$ at $t\to\infty$.

A characteristic feature of all the results discussed above is the uniform evolution of the solution to the equilibrium distribution function; in other words, "long-lived" initial distributions are absent.

Significantly poorer is set of facts concerning the case of "soft" potentials $U\sim r^{-k},\ 2<k<5.$ Here we have the results of Caflish [7], in obtaining which it was assumed that, first, there is a situation of the so-called "Grad box" with mirror-reflecting walls (i.e., the class of solutions periodic in coordinates is considered), and secondly, the initial distribution function is quite smooth and the difference $F-F_M$ decreases (in speed) exponentially fast.

2. The case of "soft" potentials

For further research, we introduce the following

Definition. Let us call by *an absolute degree of nonequilibrium of the Cauchy problem* for equation (3) the value

$$\mu=\lim_{T\to\infty}\sup_{f_0}\inf_{0\le t\le T}\left[N(f)/N(f_0)\right]$$

where f is a solution to problem (3) corresponding to the initial distribution f_0.

The result (4) therefore means that $\mu=0$.

The transition to "soft" potentials and the weakening of the conditions imposed on f_0, fundamentally change the picture of the asymptotic behavior of the solutions of equation (3).

Theorem. In the case of power-cut potential-power intermolecular interactions of the form $U\sim r^{-k}$, $2<k<5$, for each $\varepsilon>0$ and each T>0 there is an initial distribution $f_0\in L_2(x,u)$, such that for the corresponding solution $f(x,u,t)$ of problem (3) we have the inequality

$$\inf_{0\le t\le T}\left\lfloor N(f)/N(f_0)\right\rfloor>1-\varepsilon$$

Here $N(f)$ means the norm of f in $L_2(x,u)$.

Thus, $\mu=1$ and, therefore, *there exist "long-living" initial disturbances*.

The core of the proof of this theorem is the properties of the solutions of the corresponding linearized problem (designations see in [8])

$$\frac{\partial f}{\partial t} = A(f); \quad f_{t=0} = f(x,u), \qquad (5)$$

where $A \equiv -u\frac{\partial}{\partial x} + L;\ L(f) = K(f) - \nu(|u|)f$.

Lemma 1. An operator A with a domain of definition

$$D(A) = \left\{ f(x,u,t) \left| f, u_i \frac{\partial f}{\partial x_i} \in L_2(x,u) \forall t > 0 \right. \right\}$$

generates in $L_2(x,u)$ a contracting semigroup of bounded linear operators $\{T(t,),t>0\}$ of the class C_0. (The terminology corresponds to that adopted in [9]; see also chapter 1 of [6]).

Proof. The operator $A_1 = iu\frac{\partial}{\partial x}$ is self-adjoint on $D(A_1) = D(A)$ and, therefore (see [10], Sec. X.8), the operator iA_1 generates a compressive semigroup of class C_0. Since under the conditions of the theorem the operator L turns out to be bounded, self-adjoint, dissipative, $D(L) \supset D(A)$ and $\forall \phi \in D(A)$

$$\|L(\phi)\| \le a\|iA_1(\phi)\| + \|L\|\cdot\|\phi\|;\ \|\cdot\|^2 \equiv \int |\cdot|^2\, dxdu$$

for an arbitrarily small number a, then, according to the lemma from Sec. X.8 of the book [10], the operator $A = iA_1 + L$ generates a compressive semigroup $\{T(t)\}$ of class C_0, i.e. equation (4) with $f_0 \in D(A)$ has the only solution $f \in D(A)$:

$$f = T(t) f_0;\ \|T(t)\| \le 1. \qquad (6)$$

The lemma is proved.

Lemma 2. $0 \in \sigma(A)$ – spectrum of A.

Proof. We shell show that $\lambda = 0$ is the point of the essential spectrum of operator A, i.e. there is a bounded noncompact sequence $f_n \in D(A)$, satisfying condition

$$\lim_{n\to\infty}(A - \lambda E) f_n = 0.$$

Let $\varepsilon_n > 0$ and $\varepsilon_n \to 0$, monotonously decreasing; let the numbers $\xi_n > 0$ are such that $\nu(\xi_n) = \varepsilon_n$ (such ξ_n exist due to the monotonic tendency to zero collision frequencies $\nu(|u|)$ at $|u| \to \infty$ for "soft" potentials; in particular $\xi_n \to 0$, monotonously increasing). Let further Ω_n – limited area in R_u^3, located entirely outside a sphere of radius ξ_n centered at the origin and such that $\int_{\Omega_n} |u|^2\, du \le 1$.

Let $\{\nu_n(u)\}$ – sequence of functions that are finite in the domains $\Omega_n \subset R_u^3$ with media lying in their respective areas Ω_n, and orthonormalized in $L_2(u)$ (orthogonality can be achieved, for example, by choosing disjoint areas Ω_n). We put further

$$\omega_n(x) = 2^{-\frac{1}{2}}(\pi n)^{-\frac{3}{4}} \exp\left(-\frac{|x|^2}{2n}\right).$$

Note that ν_n converges weakly to zero in $L_2(u)$, and

$$\int |\omega_n(x)|^2 dx; \quad \int \left|\frac{\partial \omega_n}{\partial x_i}\right|^2 dx \xrightarrow[n\to\infty]{} 0.$$

Let $f_n = \nu_n(u)\omega_n(x)$. Obviously, $\{f_n\}$ is orthonormal sequence in $L_2(x,u)$. Let us rate

$$\|A(f_n)\| \le \left\|\nu_n u \frac{\partial \omega_n}{\partial x}\right\| + \|\nu_n \omega_n\| + \|\omega_n K(\nu_n)\|,$$

where $\|\cdot\|$ is a norm in $L_2(x,u)$. Given the choice of functions ν_n, ω_n, domains Ω_n and numbers ε_n, ξ_n we get

$$\|A(f_n)\| \to 0,\ n \to \infty.$$

Lemma 2 is proved.

Consequence. $\|T(t)\| = 1$.

Proof. By Theorem 16.3.1. and Lemma 16.3.2 from [9] we have

$$\|T(t)\| \ge \sup_{\lambda \in \sigma(A)} |\exp(t\lambda)|.$$

But according to the lemma 2, $0 \in \sigma(A)$ and therefore, $\|T(t)\| \ge 1$. On the other hand, by the lemma 1, $\|T(t)\| \le 1$, hence, $\|T(t)\| = 1$.

By virtue of (6), the statement of the main theorem is valid for the linearized equation (5). The proof of the main theorem for the nonlinear (but close to equilibrium) case is based on the technique developed in [4, 11] and the properties of solutions of equation (5) established above.

Comment. For a spatially homogeneous linearized equation, the formulation of the main result will slightly change, namely, for the semigroup $T(t) = \exp(tL)$ generated by a bounded (for "soft" potentials) operator L, there is a presentation

$$T(t) = \int_{\sigma(L)\setminus\{0\}} e^{t\lambda} dE_\lambda + \sum_{j=0}^{4} P_{\psi_j},$$

where P_{ψ_j} are projectors on one-dimensional subspaces of additive invariants ψ_j.

Denote the first term on the right-hand side of the last equality by $T_\perp(t)$. Then

$$f = T_\perp(t) f_0 + \sum_{j=1}^{4} a_j \psi_j.$$

Similarly, to the above, it is easy to show that $\|T_\perp(t)\| = 1$.

3. Conclusions

The work is devoted to the study of the stability of solutions of the linearized Boltzmann equation in the case of “soft” intermolecular interaction potentials (that is, power potentials with exponents less than 5). In this case, a fact of loss of stability of solutions is found that is very curious from a physical point of view: it turns out that there are initial perturbations that "live" for an arbitrarily long time!

Recall that for “hard” potentials (exponent greater than 5) this fact does not occur. We also note that the potential of Coulomb interactions is “soft”.

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