A study of convergence of ξ approximations transform by region depended given as determined by $\sigma_n(t)$ Functions on entire complex plane

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Abstract: *In this study, the convergence behavior of the ξ approximants transform given as determined by* $\sigma_n(t)$ *functions for the exponential operators is investigated on the entire complex plain. An algorithm is developed to observe how to transform a initial region on the complex plane defined by ξ approximants.*
Keywords*: LIE OPERATORS, COMPLEX RECURSIVE FUNCTIONS, EVALUATION OPERATORS*

1. Introduction

In this study, the convergence behavior of the ξ approximants transform given as determined by $\sigma_n(t)$ functions for the exponential operators is investigated on the entire complex plain and an algorithm is developed to observe how to transform a initial region on the complex plane defined by ξ approximants.

System with n degrees of freedom will be characterized by $x_1,\ x_2, \dots, x_n$ complex variables, which are considered as the coordinates of a point or components of a vector in an n-dimensional complex vector space.

$\{\, x_1(t), x_2(t), \dots, x_n(t)\}$: System

$x_1,\ x_2, \dots, x_n$: Phase space vector components in this system.

Hence $Q\left(t_f, t_i\right)$ global evolution operator is defined as

$$x\left(t_f\right) = \ Q\left(t_f, t_i\right).x\left(t_i\right)$$

where t_i and t_f denote the initial and final states respectively. If factorization of evolution operators is considered as a sequence of rather simple global evolution operators then the evolution operator can be written as follows

$$Q\left(t_f, t_i\right) = e^{\left(t_f - t_i\right).\mathrm{S}}$$

where S is defined through

$$S = \sum_{j=1}^{N} f_j\ (x_1,\ x_2, \dots, x_N)\frac{\partial}{\partial x_j}$$

so Q evolution operator can be assumed to be written as

$$Q = e^{tf(x)\frac{\partial}{\partial x}} = \prod_{j=1}^{\infty} e^{\sigma_j(t)x^j\frac{\partial}{\partial x}} \qquad \sigma_j(t) = tf_j$$

where $L = f(x)\frac{\partial}{\partial x}$ and

$$f(x) = \sum_{j=1}^{\infty} f_j\ x^j \qquad |x| < \rho$$

This equation is factorization formula for the one-dimensional case.

The essential approximation is to truncate

$$Q = \prod_{j=1}^{\infty} e^{\sigma_j(t)x^j\frac{\partial}{\partial x}} \qquad \sigma_j(t) = t.f_j$$

to a finite order. By this way it produces the following approximation.

$$\overline{\xi_n}(x,t) = \left\{\prod_{j=1}^{n} Q^{(j)}\right\} x$$

If the infinite product representation of Q converges then the following result can be obtained:

$$\bar{\xi}(x,t) = Qx = e^{tf(x)\frac{\partial}{\partial x}}\, x = \lim_{n\to\infty} \overline{\xi_n}$$

A recursion relation for these approximants can be shown as follows:

$$\overline{\xi_{n+1}} = \frac{\overline{\xi_n}(x,t)}{\left[1 - n\sigma_{n+1}(t)\,\overline{\xi_n}^{\,n}(x,t)\right]^{\frac{1}{n}}}$$

And this is a recursion relation with an initial member evaluated as follows:

$$\bar{\xi_1}(x,t) = e^{\sigma_1(t)x\frac{\partial}{\partial x}}\ x = xe^{\sigma_1(t)} = xe^{f_1 t}$$

Although this is a simple recursion relation, the existence of f_1 may not be suitable for numerical purpose depending on the value of f_1. So we can normalize ξ- approximants as follows:

$$\overline{\sigma_{n+1}} = n\sigma_{n+1}e^{nf_1 t}$$

Then the final recursion relation becomes as follows:

$$\xi_{n+1}(x,t) = \frac{\xi_n(x,t)}{\left(1 - \overline{\sigma_{n+1}}(t)\xi_n^n(x,t)\right)^{\frac{1}{n}}}$$

The relation between the final and the previous approximants can be given as

$$\overline{\xi_n}(x,t) = \xi_n(x,t)\ xe^{f_1 t}$$

In this study the convergence of the ξ-approximant sequences in the complex plane is main issue. The above transformation of ξ_n to ξ_{n+1} can be interpreted as applying some basic elementary transformations consecutively

$$\xi_{n+1} = \frac{\xi_n}{(1 - \sigma_{n+1}\xi_n^n)^{\frac{1}{n}}}$$

Here $\sigma_n(t)$ functions are assumed to be given and the ξ-approximants' nature are determined by these $\sigma_n(t)$ functions. The main purpose of the examination of this convergence is to give important information about the convergence of finite product sequences that appears during the factorization of these sequences.

In this study an algorithm is developed to find out or observe how the above recursion relation defined via ξ-approximants transform a given initial region on the complex plane defined by ξ-approximants and a develop a computer program based on this algorithm.

A singular point taken in the domain can carry us to infinity on the ξ_{n+1}–plane as can be noticed through

$$\xi_{n+1} = \frac{\xi_n}{(1 - \sigma_{n+1}\xi_n^n)^{\frac{1}{n}}}$$

Therefore one can obtain a singularity free initial region on ξ_n – plane by determining the locations of these singularities and discarding them from the domain.

In this study, the original contribution is the separation of the recursion relation between two consecutive ξ-approximants into basic simple consecutive transformations.

The other contributions are the construction of an algorithm that evaluates the region variations through the consecutive transformations and to develop a computer program to execute this algorithm. The computer program is written in C++ language and Mathematica is used for graphics.

2. Examining of The Convergence

If we assume that the functions represented by $\sigma_n(t)$ are given quantities, we need to trace how the ξ-approximants behave on the complex number plane.

$$\xi_{n+1} = \frac{\xi_n}{(1-\sigma_{n+1}\xi_n^n)^{\frac{1}{n}}}$$

Although it is important that the functions $\sigma_n(t)$ depend on t, it will be assumed that the $\sigma_n(t)$ take into for a certain value of t, that is, we will deal with pointwise convergence.

Let us consider $\sigma_{n+1}(t)$ function sequences as constant, but convergent or non-convergent sequences.

For $|\xi_1|$=R circle, we obtain iterative functions as below.

$$\xi_1 = \xi_1$$

$$\xi_2 = \frac{\xi_1}{1-\sigma_2\xi_1}$$

$$\xi_3 = \frac{\xi_2}{(1-\sigma_3\xi_2^2)^{\frac{1}{2}}}$$

$$\xi_4 = \frac{\xi_3}{(1-\sigma_4\xi_3^2)^{\frac{1}{3}}}$$

$$\vdots$$

If the sequences $\sigma_1, \sigma_2, \sigma_3, \cdots, \sigma_n$ are taken as constant arrays such as $1,\ n,\ \frac{1}{n},\ \frac{1}{n^2}\ \ldots$ ξ-approximants will show different behavior.

It is impossible to solve the ξ_{n+1} sequences by analytically except by taking $n = 1$, for $|z| = R$ and $\sigma = const.$

However, it may be possible to examine the graphs found by some operations on functions.

The graphs in Figure 1 show four consecutive transformations, taking the $\sigma_n(t) = n$ sequence of the $|\xi_1| = 1$ unit circle in the complex number plane.

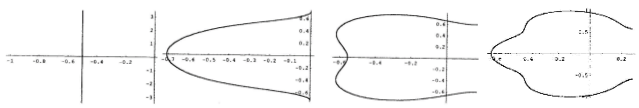

Figure 1

Here it can be easily seen that the transformation obtained with ξ_2 transforms into the line $u = -\frac{1}{2}$, and the inside of the unit circle transforms into the region to the right of the line $u = -\frac{1}{2}$.

In the next step, the line $u = -\frac{1}{2}$ turn into a parabola with its peak around $x = -0.7$ and its arms extending to $x = +\infty$.

The region to the right of $u = -\frac{1}{2}$ has turned inside of the parabola. The arms of the parabola cut the y-axis around ± 0.7.

In the next transformations, generally appear look likes these parabolas.

However, they are not smooth due to error congestion.

The graphs in Figure 2 show four consecutive transformations, taking the $\sigma_n(t) = 1$ sequence of the $|\xi_1| = 1$ unit circle in the complex number plane.

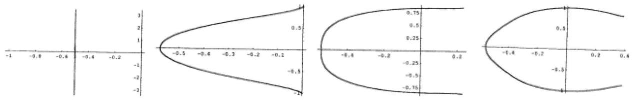

Figure 2

The graphs in Figure 3 show four consecutive transformations, taking the $\sigma_n(t) = \frac{1}{n}$ sequence of the $|\xi_1| = 1$ unit circle in the complex number plane.

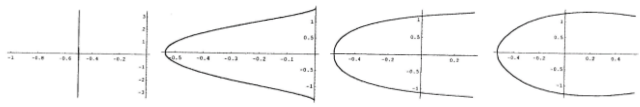

Figure 3

3. Conclusion

In this study has shown for the $\sigma_n(t)$ sequences that the equivalent in the ξ_n-plane of a circular region without a singularity for the ξ_1-plane will remain without singularity in the ξ_n series, regardless of n and time.

Here all the transformations remain open at infinity because they are the branch points of systems.

4. References

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