Formulation of axisymmetric boundary value problems of the linear theory of elasticity for canonical bodies in harmonic potentials

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Abstract: *The paper is based on the representation of the fundamental solution of the linear elasticity theory of the mechanics of a deformable solid in the J. Dougall's form through spatial harmonic functions. The axisymmetric problem of the elasticity theory in a cylindrical coordinate system for bodies bounded by a canonical surface is formulated. As a case, the boundary value problem of pure torsion is formulated and the elastic characteristics and structure of the corresponding external loads on the side surface of a given isotropic elastic body in the above-mentioned harmonic potentials are presented. This approach makes it possible to obtain and extend the set of exact analytical solutions of boundary value problems of the spatial elasticity theory and is the theoretical basis for calculating the strength parameters of mechanical systems.*

Keywords: *THREE-DIMENSIONAL THEORY OF ELASTICITY; BOUNDARY VALUE PROBLEM; DISPLACEMENT VECTOR; STRESS TENSOR; DEFORMATION TENSOR; BASIC SOLUTIONS; HARMONIC FUNCTION.*

1. Introduction

An important area of research in the mechanics of a deformed body is the spatial problems of the elasticity theory. The relevance of basic research on this topic is characterized by the fact that in most cases, the stress-strain state of the body is 3-dimensional. Today, both numerical and analytical approaches for solving boundary value problems of the spatial elasticity theory have been built.

Methods of solving the problems of static elasticity known from the literature are based on solutions of homogeneous equilibrium equations in displacements using harmonic and biharmonic functions. Such solutions have been proposed, in particular, by W.Kelvin, P.Tait [1] and M.Boussinesq [2]. B.Galerkin [3] presented the general solution of the equations of elastic equilibrium of an isotropic body through three biharmonic functions. In the works of P.F.Papkovych [4] and H.Neuber [5] the form of the general solution was given through vector and scalar harmonic functions. Optimization of this representation only through three spatial harmonic functions have been published in scientific papers [6,7]. This solution representation has become the basis for solving the three-dimensional boundary value problem of the theory of elasticity for bodies of rotation [8].

Given the complexity of constructing solutions of boundary value problems of the elasticity theory in three-dimensional space, assumptions are often made about the structure of deformation of elastic bodies. The monographs of A.I. Lurie [9] and J.N.Goodier, S.P.Timoshenko [10] investigates general methods for solving the basic equations of equilibrium of an isotropic and anisotropic body, which is of great importance for the development of research on spatial problems of the elasticity theory.

J.Dougall [11] was one of the first to propose an approach to construct a general solution to the problem of the theory of elasticity for a cylinder in three-dimensional space. In [12], the authors used two representations of Dugall through harmonic functions to construct a general solution of the problem of the elasticity theory for a continuous finite cylinder, which allows to satisfy the boundary conditions both at the ends and on the side surface of the cylinder.

Representation of the Dugall's solution through three spatial harmonic scalar functions was used as a basis for formulating the boundary value problem of torsion for bodies bounded by a canonical surface, which will expand the set of exact analytical solutions and become a theoretical basis for calculating the strength parameters of mechanical systems.

2. Formulation of the problem of the linear elasticity theory for a body bounded by a canonical surface

An isotropic elastic body X in three-dimensional space is considered, which is bounded by the surface ∂X, which is described by the canonical curve $f(r,\theta,z)=0$. (Fig. 1).

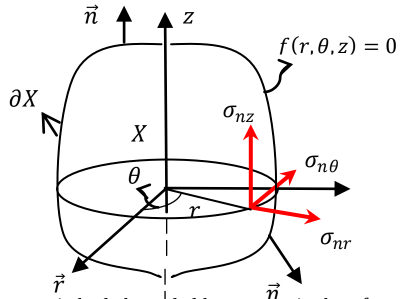

Fig. 1 *Elastic isotropic body bounded by a canonical surface*

The body is under the action of a stationary force load, which is applied to the lateral surface of the body ∂X.

The first boundary value problem of the linear elasticity theory for a given elastic isotropic body in the absence of bulk forces in a cylindrical coordinate system (r,θ,z) has the form.

− equilibrium equation:

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{\partial\sigma_{rz}}{\partial z}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}=0,$$

$$(1)\qquad \frac{\partial\sigma_{rz}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta z}}{\partial\theta}+\frac{\partial\sigma_{zz}}{\partial z}+\frac{\sigma_{rz}}{r}=$$

$$\frac{\partial\sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{\partial\sigma_{\theta z}}{\partial z}+\frac{2\sigma_{r\theta}}{r}=0.$$

− stress tensor $\hat{\sigma}(r,\theta,z)$ satisfies the boundary conditions:

$$(2)\qquad \vec{\sigma}_n\equiv(\vec{n}\cdot\hat{\sigma})|_{\partial X}=\vec{\sigma}_n^{+},$$

where $\vec{\sigma}_n(r,\theta,z)$ – the stress vector on the surface ∂X; $\vec{\sigma}_n^{+}(r,\theta,z)$ – a given vector of surface forces on ∂X; $\vec{n}\equiv n_r\vec{e}_z+n_\theta\vec{e}_\theta+n_z\vec{e}_z=\vec{\nabla}f(r,\theta,z)/|\vec{\nabla}f(r,\theta,z)|$ – external normal to the body surface ∂X.

In a cylindrical coordinate system, the components of the stress tensor $\hat{\sigma}$ are fed through the deformation tensor $\hat{\varepsilon}$ by the following relationship:

$$\sigma_{rr}=\lambda\vartheta+2\mu\varepsilon_{rr},\quad \sigma_{\theta\theta}=\lambda\vartheta+2\mu\varepsilon_{\theta\theta},\quad \sigma_{zz}=\lambda\vartheta+2\mu\varepsilon_{zz},$$

$$(3)\qquad \sigma_{rz}=\mu\varepsilon_{rz},\quad \sigma_{r\theta}=\mu\varepsilon_{r\theta},\quad \sigma_{\theta z}=\mu\varepsilon_{\theta z},$$

where $\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{zz}, \varepsilon_{rz}, \varepsilon_{r\theta}, \varepsilon_{\theta z}$ are the components of the deformation tensor $\hat{\varepsilon}$; μ, λ - elastic Lamé constants.

The deformation tensor $\hat{\varepsilon}$ is expressed in terms of the components of the displacement vector $\vec{u}(u_r, u_\theta, u_z)$ as follows:

$$(4)\quad \varepsilon_{rr} = \frac{\partial U_r}{\partial r}, \qquad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_r}{r}, \qquad \varepsilon_{zz} = \frac{\partial U_z}{\partial z},$$

$$\varepsilon_{rz} = \frac{\partial U_r}{\partial z} + \frac{\partial U_z}{\partial r}, \varepsilon_{r\theta} = \frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r} + \frac{1}{r}\frac{\partial U_r}{\partial \theta}, \varepsilon_{\theta z} = \frac{1}{r}\frac{\partial U_z}{\partial \theta} + \frac{\partial U_\theta}{\partial z}.$$

Volumetric deformation ϑ is calculated by the formula:

$$(5)\qquad \vartheta = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz} = \frac{\partial U_r}{\partial r} + \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_r}{r} + \frac{\partial U_z}{\partial z}.$$

3.1. Formulation of the boundary value problem of the elasticity theory in Dugall's harmonic potential

Consider the representation of the components of the displacement vector $\vec{u}$ proposed by J.Dugall [11] through three harmonic spatial functions $\varphi(r,\theta,z), \omega(r,\theta,z), \psi(r,\theta,z)$, which was in among the first approaches to construct the solution of equilibrium equations (1) of the linear elasticity theory in displacements in three-dimensional space:

$$U_r = 2z\frac{\partial^2\varphi}{\partial r\partial z} + (3-4v)\frac{\partial\varphi}{\partial r} + \frac{\partial\omega}{\partial r} + \frac{2}{r}\frac{\partial\psi}{\partial\theta},$$

$$(6)\qquad U_\theta = \frac{2z}{r}\frac{\partial^2\varphi}{\partial\theta\partial z} + \frac{3-4v}{r}\frac{\partial\varphi}{\partial\theta} + \frac{1}{r}\frac{\partial\omega}{\partial\theta} - 2\frac{\partial\psi}{\partial r},$$

$$U_z = 2z\frac{\partial^2\varphi}{\partial z^2} - (3-4v)\frac{\partial\varphi}{\partial z} + \frac{\partial\omega}{\partial z}.$$

where v - the Poisson's ratio.

Substituting the components of the displacement vector (6) in the ratio of the deformation tensor (4), we obtain:

$$\varepsilon_{rr} = 2z\frac{\partial^3\varphi}{\partial r^2\partial z} + (3-4v)\frac{\partial^2\varphi}{\partial r^2} + \frac{\partial^2\omega}{\partial r^2} + \frac{2}{r}\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{2}{r^2}\frac{\partial\psi}{\partial\theta},$$

$$\varepsilon_{\theta\theta} = \frac{2z}{r^2}\frac{\partial^3\varphi}{\partial\theta^2\partial z} + \frac{3-4v}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{2z}{r}\frac{\partial^2\varphi}{\partial r\partial z} + \frac{(3-4v)}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\omega}{\partial\theta^2} + \frac{1}{r}\frac{\partial\omega}{\partial r} - \frac{2}{r}\frac{\partial^2\psi}{\partial\theta\partial r} + \frac{2}{r^2}\frac{\partial\psi}{\partial\theta},$$

$$\varepsilon_{zz} = 2z\frac{\partial^3\varphi}{\partial z^3} - (1-4v)\frac{\partial^2\varphi}{\partial z^2} + \frac{\partial^2\omega}{\partial z^2},$$

$$\varepsilon_{rz} = 2\left(2z\frac{\partial^3\varphi}{\partial r\partial z^2} + \frac{\partial^2\varphi}{\partial r\partial z} + \frac{\partial^2\omega}{\partial r\partial z} + \frac{1}{r}\frac{\partial^2\psi}{\partial\theta\partial z}\right),$$

$$\varepsilon_{r\theta} = 2\left(\frac{2z}{r}\frac{\partial^3\varphi}{\partial r\partial\theta\partial z} + \frac{3-4v}{r}\frac{\partial^2\varphi}{\partial r\partial\theta} - \frac{2z}{r^2}\frac{\partial^2\varphi}{\partial\theta\partial z} - \frac{3-4v}{r^2}\frac{\partial\varphi}{\partial\theta} + \frac{1}{r}\frac{\partial^2\omega}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\omega}{\partial\theta} - 2\frac{\partial^2\psi}{\partial r^2} + \frac{\partial^2\psi}{\partial z^2}\right),$$

$$(7)\quad \varepsilon_{\theta z} = 2\left(\frac{2z}{r}\frac{\partial^3\varphi}{\partial\theta\partial z^2} + \frac{1}{r}\frac{\partial^2\varphi}{\partial\theta\partial z} + \frac{1}{r}\frac{\partial^2\omega}{\partial\theta\partial z} - \frac{\partial^2\psi}{\partial r\partial z}\right).$$

Accordingly, the volume deformation ϑ under the conditions of harmony of the functions $\varphi(r,\theta,z)$, $\omega(r,\theta,z)$, $\psi(r,\theta,z)$ is calculated as follows:

$$(8)\qquad \vartheta = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz} = -4(1-2v)\frac{\partial^2\varphi}{\partial z^2}.$$

Based on relations (7) and (8) we find the form of the components of the stress tensor (3):

$$\sigma_{rr} = 2\mu\left(2z\frac{\partial^3\varphi}{\partial r^2\partial z} + (3-4v)\frac{\partial^2\varphi}{\partial r^2} - 4v\frac{\partial^2\varphi}{\partial z^2} + \frac{\partial^2\omega}{\partial r^2} + \frac{2}{r}\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{2}{r^2}\frac{\partial\psi}{\partial\theta}\right),$$

$$\sigma_{\theta\theta} = 2\mu\left(\frac{2z}{r^2}\frac{\partial^3\varphi}{\partial\theta^2\partial z} + \frac{3-4v}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} - 4v\frac{\partial^2\varphi}{\partial z^2} + \frac{2z}{r}\frac{\partial^2\varphi}{\partial r\partial z} + \frac{(3-4v)}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\omega}{\partial\theta^2} + \frac{1}{r}\frac{\partial\omega}{\partial r} - \frac{2}{r}\frac{\partial^2\psi}{\partial\theta\partial r} + \frac{2}{r^2}\frac{\partial\psi}{\partial\theta}\right),$$

$$\sigma_{zz} = 2\mu\left(2z\frac{\partial^3\varphi}{\partial z^3} - \frac{\partial^2\varphi}{\partial z^2} + \frac{\partial^2\omega}{\partial z^2}\right),$$

$$\sigma_{rz} = 2\mu\left(2z\frac{\partial^3\varphi}{\partial r\partial z^2} + \frac{\partial^2\varphi}{\partial r\partial z} + \frac{\partial^2\omega}{\partial r\partial z} + \frac{1}{r}\frac{\partial^2\psi}{\partial\theta\partial z}\right),$$

$$\sigma_{r\theta} = 2\mu\left(\frac{2z}{r}\frac{\partial^3\varphi}{\partial r\partial\theta\partial z} + \frac{3-4v}{r}\frac{\partial^2\varphi}{\partial r\partial\theta} - \frac{2z}{r^2}\frac{\partial^2\varphi}{\partial\theta\partial z} - \frac{3-4v}{r^2}\frac{\partial\varphi}{\partial\theta} + \frac{1}{r}\frac{\partial^2\omega}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\omega}{\partial\theta} - 2\frac{\partial^2\psi}{\partial r^2} - \frac{\partial^2\psi}{\partial z^2}\right),$$

$$(9)\quad \sigma_{\theta z} = 2\mu\left(\frac{2z}{r}\frac{\partial^3\varphi}{\partial\theta\partial z^2} + \frac{1}{r}\frac{\partial^2\varphi}{\partial\theta\partial z} + \frac{1}{r}\frac{\partial^2\omega}{\partial\theta\partial z} - \frac{\partial^2\psi}{\partial r\partial z}\right).$$

Based on the normal components of the stress tensor $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}$ we find its first invariant:

$$(10)\qquad J = \sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz} = -8\mu(v+1)\frac{\partial^2\varphi}{\partial z^2}.$$

Thus, using the representation of the fundamental solution of equilibrium equations in the Dugall's form, the boundary value problem of the linear theory of elasticity (1) - (2) is reformulated into a boundary value problem for harmonic scalar potentials:

$$(11)\qquad \Delta\big(\varphi(r,\theta,z), \omega(r,\theta,z), \psi(r,\theta,z)\big) = 0,$$

which satisfy the corresponding boundary conditions on the surface ∂X:

$$(\vec{n}\cdot\hat{\sigma})|_{\partial X} = \big((n_r\sigma_{rr} + n_\theta\sigma_{r\theta} + n_z\sigma_{rz})\overrightarrow{e_r} + (n_r\sigma_{r\theta} + n_\theta\sigma_{\theta\theta} + n_z\sigma_{\theta z})\overrightarrow{e_\theta} +$$

$$(12)\qquad + (n_r\sigma_{rz} + n_\theta\sigma_{\theta z} + n_z\sigma_{zz})\overrightarrow{e_z}\big)|_{\partial X} = \vec{\sigma}_n^+,$$

where $\Delta \equiv \vec{\nabla}\cdot\vec{\nabla} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$ – Laplace operator. $\vec{\nabla} = \frac{\partial}{\partial r}\overrightarrow{e_r} + \frac{1}{r}\frac{\partial}{\partial\theta}\overrightarrow{e_\theta} + \frac{\partial}{\partial z}\overrightarrow{e_z}$ – Hamilton operator.

3.2. Statement of axisymmetric problems of the elasticity theory in harmonic potentials

Let the axis of symmetry of the body coincide with the axis z of the cylindrical coordinate system (r,θ,z). Given that the problem is axisymmetric, the displacement vector $\vec{u}$ does not depend on the angle θ:

$$(13)\qquad \frac{\partial\vec{u}}{\partial\theta} = 0.$$

Given this ratio, the following conditions are imposed on the components of the displacement vector $\vec{u}$ in the Dugall's form (6):

$$\frac{\partial U_r}{\partial\theta} = 2z\frac{\partial^3\varphi}{\partial r\partial\theta\partial z} + (3-4v)\frac{\partial^2\varphi}{\partial r\partial\theta} + \frac{\partial^2\omega}{\partial r\partial\theta} + \frac{2}{r}\frac{\partial^2\psi}{\partial\theta^2} = 0,$$

$$(14)\quad \frac{\partial U_\theta}{\partial\theta} = \frac{2z}{r}\frac{\partial^3\varphi}{\partial\theta^2\partial z} + \frac{3-4v}{r}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{r}\frac{\partial^2\omega}{\partial\theta^2} - 2\frac{\partial^2\psi}{\partial r\partial\theta} = 0,$$

$$\frac{\partial U_z}{\partial\theta} = 2z\frac{\partial^3\varphi}{\partial\theta\partial z^2} - (3-4v)\frac{\partial^2\varphi}{\partial\theta\partial z} + \frac{\partial^2\omega}{\partial\theta\partial z} = 0.$$

From these conditions it follows that the harmonic functions $\varphi(r,\theta,z)$, $\omega(r,\theta,z)$ and $\psi(r,\theta,z)$ are linear functions on the variable θ, and we also get some elms on these functions, in particular:

$$\frac{\partial^2\omega}{\partial r\partial\theta} = -2z\frac{\partial^3\varphi}{\partial r\partial\theta\partial z} - (3-4v)\frac{\partial^2\varphi}{\partial r\partial\theta},$$
$$(15)\quad \frac{\partial^2\psi}{\partial r\partial\theta} = 0,$$
$$\frac{\partial^2\omega}{\partial\theta\partial z} = -2z\frac{\partial^3\varphi}{\partial\theta\partial z^2} + (3-4v)\frac{\partial^2\varphi}{\partial\theta\partial z}.$$

Accordingly, the components of the deformation tensor $\hat{\varepsilon}$ (7) take the form:

$$\varepsilon_{rr} = 2z\frac{\partial^3\varphi}{\partial r^2\partial z} + (3-4v)\frac{\partial^2\varphi}{\partial r^2} + \frac{\partial^2\omega}{\partial r^2} - \frac{2}{r^2}\frac{\partial\psi}{\partial\theta},$$
$$\varepsilon_{\theta\theta} = \frac{2z}{r}\frac{\partial^2\varphi}{\partial r\partial z} + \frac{3-4v}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r}\frac{\partial\omega}{\partial r} + \frac{2}{r^2}\frac{\partial\psi}{\partial\theta},$$
$$\varepsilon_{zz} = 2z\frac{\partial^3\varphi}{\partial z^3} - (1-4v)\frac{\partial^2\varphi}{\partial z^2} + \frac{\partial^2\omega}{\partial z^2},$$
$$\varepsilon_{rz} = 2\left(2z\frac{\partial^3\varphi}{\partial r\partial z^2} + \frac{\partial^2\varphi}{\partial r\partial z} + \frac{\partial^2\omega}{\partial r\partial z} + \frac{1}{r}\frac{\partial^2\psi}{\partial\theta\partial z}\right),$$
$$\varepsilon_{r\theta} = 2\left(-\frac{2z}{r^2}\frac{\partial^2\varphi}{\partial\theta\partial z} - \frac{3-4v}{r^2}\frac{\partial\varphi}{\partial\theta} - \frac{1}{r^2}\frac{\partial\omega}{\partial\theta} - \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r}\right),$$
$$(16)\quad \varepsilon_{\theta z} = 2\left(\frac{4(1-v)}{r}\frac{\partial^2\varphi}{\partial\theta\partial z} - \frac{\partial^2\psi}{\partial r\partial z}\right).$$

The components of the stress tensor $\hat{\sigma}$, according to (16), are as follows:

$$\sigma_{rr} = 2\mu\left(2z\frac{\partial^3\varphi}{\partial r^2\partial z} + (3-4v)\frac{\partial^2\varphi}{\partial r^2} - 4v\frac{\partial^2\varphi}{\partial z^2} + \frac{\partial^2\omega}{\partial r^2} - \frac{2}{r^2}\frac{\partial\psi}{\partial\theta}\right),$$
$$\sigma_{\theta\theta} = 2\mu\left(\frac{2z}{r}\frac{\partial^2\varphi}{\partial r\partial z} - 4v\frac{\partial^2\varphi}{\partial z^2} + \frac{3-4v}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r}\frac{\partial\omega}{\partial r} + \frac{2}{r^2}\frac{\partial\psi}{\partial\theta}\right),$$
$$\sigma_{zz} = 2\mu\left(2z\frac{\partial^3\varphi}{\partial z^3} - \frac{\partial^2\varphi}{\partial z^2} + \frac{\partial^2\omega}{\partial z^2}\right),$$
$$\sigma_{rz} = 2\mu\left(2z\frac{\partial^3\varphi}{\partial r\partial z^2} + \frac{\partial^2\varphi}{\partial r\partial z} + \frac{\partial^2\omega}{\partial r\partial z} + \frac{1}{r}\frac{\partial^2\psi}{\partial\theta\partial z}\right),$$
$$\sigma_{r\theta} = 2\mu\left(-\frac{2z}{r^2}\frac{\partial^2\varphi}{\partial\theta\partial z} - \frac{3-4v}{r^2}\frac{\partial\varphi}{\partial\theta} - \frac{1}{r^2}\frac{\partial\omega}{\partial\theta} - \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r}\right),$$
$$(17)\quad \sigma_{\theta z} = 2\mu\left(\frac{4(1-v)}{r}\frac{\partial^2\varphi}{\partial\theta\partial z} - \frac{\partial^2\psi}{\partial r\partial z}\right).$$

Based on the elms (15), the first invariant of the stress tensor is expressed only through one harmonic scalar function $\varphi(r,\theta,z)$:

$$(18)\quad J = \sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz} = -8\mu(v+1)\frac{\partial^2\varphi}{\partial z^2}.$$

Therefore, the axisymmetric boundary value problem on the harmonic potentials $\varphi(r,\theta,z)$**,** $\omega(r,\theta,z)$ and $\psi(r,\theta,z)$ has the form:

$$(19)\quad \Delta\big(\varphi(r,\theta,z),\omega(r,\theta,z),\psi(r,\theta,z)\big) = 0,$$

which satisfy the corresponding boundary conditions on the body surface ∂X:

$$(\vec{n}\cdot\hat{\sigma})|_{\partial X} = \big((n_r\sigma_{rr} + n_\theta\sigma_{r\theta} + n_z\sigma_{rz})\vec{e_r} +$$
$$+(n_r\sigma_{r\theta} + n_\theta\sigma_{\theta\theta} + n_z\sigma_{\theta z})\vec{e_\theta} +$$
$$(20)\quad + (n_r\sigma_{rz} + n_\theta\sigma_{\theta z} + n_z\sigma_{zz})\vec{e_z}\big)|_{\partial X} = \vec{\sigma}_n^{+},$$

where the components of the stress tensor $\sigma_{rr},\sigma_{\theta\theta},\sigma_{zz},\sigma_{r\theta},\sigma_{rz},\sigma_{\theta z}$ are given by the relations (17).

3.3. Statement of the boundary value problem of pure torsion

Consider one of the axisymmetric stress states of the body of rotation - pure torsion, which is described when the conditions $U_r = U_z = 0$:

$$U_r = 2z\frac{\partial^2\varphi}{\partial r\partial z} + (3-4v)\frac{\partial\varphi}{\partial r} + \frac{\partial\omega}{\partial r} + \frac{2}{r}\frac{\partial\psi}{\partial\theta} = 0,$$
$$U_\theta = \frac{2z}{r}\frac{\partial^2\varphi}{\partial\theta\partial z} + \frac{3-4v}{r}\frac{\partial\varphi}{\partial\theta} + \frac{1}{r}\frac{\partial\omega}{\partial\theta} - 2\frac{\partial\psi}{\partial r},$$
$$(20)\quad U_z = 2z\frac{\partial^2\varphi}{\partial z^2} - (3-4v)\frac{\partial\varphi}{\partial z} + \frac{\partial\omega}{\partial z} = 0.$$

Given the relation (20), the components of the deformation tensor $\hat{\varepsilon}$ (16) and, accordingly, its first invariant ϑ take the form:

$$\varepsilon_{rr} = 2z\frac{\partial^3\varphi}{\partial r^2\partial z} + (3-4v)\frac{\partial^2\varphi}{\partial r^2} + \frac{\partial^2\omega}{\partial r^2} - \frac{2}{r^2}\frac{\partial\psi}{\partial\theta} = 0,$$
$$\varepsilon_{\theta\theta} = \frac{2z}{r}\frac{\partial^2\varphi}{\partial r\partial z} + \frac{3-4v}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r}\frac{\partial\omega}{\partial r} + \frac{2}{r^2}\frac{\partial\psi}{\partial\theta} = 0,$$
$$\varepsilon_{zz} = 2z\frac{\partial^3\varphi}{\partial z^3} - (1-4v)\frac{\partial^2\varphi}{\partial z^2} + \frac{\partial^2\omega}{\partial z^2} = 0,$$
$$\varepsilon_{rz} = 2\left(2z\frac{\partial^3\varphi}{\partial r\partial z^2} + \frac{\partial^2\varphi}{\partial r\partial z} + \frac{\partial^2\omega}{\partial r\partial z} + \frac{1}{r}\frac{\partial^2\psi}{\partial\theta\partial z}\right) = 0,$$
$$\varepsilon_{r\theta} = 2\left(-\frac{2z}{r^2}\frac{\partial^2\varphi}{\partial\theta\partial z} - \frac{3-4v}{r^2}\frac{\partial\varphi}{\partial\theta} - \frac{1}{r^2}\frac{\partial\omega}{\partial\theta} - \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r}\right)$$
$$(21)\quad \varepsilon_{\theta z} = 2\left(\frac{4(1-v)}{r}\frac{\partial^2\varphi}{\partial\theta\partial z} - \frac{\partial^2\psi}{\partial r\partial z}\right).$$

$$(22)\quad \vartheta = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz} = -4(1-2v)\frac{\partial^2\varphi}{\partial z^2} = 0.$$

Based on relations (21)-(22), the components of the stress tensor $\hat{\sigma}$ are written as follows:

$$\sigma_{rr} = 0,\quad \sigma_{\theta\theta} = 0,\quad \sigma_{zz} = 0,\quad \sigma_{rz} = 0,$$
$$\sigma_{r\theta} = 2\mu\left(-\frac{2z}{r^2}\frac{\partial^2\varphi}{\partial\theta\partial z} - \frac{3-4v}{r^2}\frac{\partial\varphi}{\partial\theta} - \frac{1}{r^2}\frac{\partial\omega}{\partial\theta} - \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r}\right),$$
$$(23)\quad \sigma_{\theta z} = 2\mu\left(\frac{4(1-v)}{r}\frac{\partial^2\varphi}{\partial\theta\partial z} - \frac{\partial^2\psi}{\partial r\partial z}\right).$$

and, accordingly, the first invariant J is identically equal to zero.

Under the axisymmetric problem (15) and the torsion problem (21) - (22) we write the relation:

$$\frac{\partial^2\psi}{\partial r\partial\theta} = 0,\qquad \frac{\partial^2\varphi}{\partial z^2} = 0,$$
$$(24)\quad 2z\frac{\partial^3\varphi}{\partial z^3} - (1-4v)\frac{\partial^2\varphi}{\partial z^2} + \frac{\partial^2\omega}{\partial z^2} = 0.$$

on the basis of which we obtain the equations for harmonic functions:

$$(25)\quad \frac{\partial^2\psi}{\partial r\partial\theta} = 0,\quad \frac{\partial^2\omega}{\partial z^2} = 0,\quad \frac{\partial^2\varphi}{\partial z^2} = 0.$$

Under these conditions, we present the structure of harmonic potentials $\varphi(r,\theta,z)$, $\omega(r,\theta,z)$ and $\psi(r,\theta,z)$ in the form:

$$\psi(r,\theta,z) = \psi_0(r,z) + \psi_1(\theta,z) + \psi_2(z),$$
$$\varphi(r,\theta,z) = \varphi_0(r,\theta) + z\varphi_1(r,\theta),$$
$$(26)\quad \omega(r,\theta,z) = \omega_0(r,\theta) + z\omega_1(r,\theta).$$

where $\psi_i, \varphi_j, \omega_j, (i=\overline{0,2})(j=\overline{0,1})$ – unknown harmonic functions.

Note that $\psi_1(\theta,z)$ and $\psi_2(z)$ do not affect the structure of the tangential components of the stress tensor $\sigma_{r\theta}$ and $\sigma_{\theta z}$, so they can be considered equal to zero.

4. Results and discussion

The obtained structure of harmonic functions (26) is substituted into the initial representation of the displacement vector $\vec{u}$ (20) for the pure torsion problem. We obtain finite expressions and connections for unknown harmonic functions $\psi_0(r,z)$, $\varphi_0(r,\theta)$, $\varphi_1(r,\theta)$, $\omega_0(r,\theta)$, $\omega_1(r,\theta)$:

$$U_r = \frac{\partial}{\partial r}\left((3-4v)\varphi_0 + 8(1-v)z\varphi_1 + \omega_0\right) = 0,$$

$$U_\theta = \frac{\partial}{\partial \theta}\left[\frac{1}{r}\left((3-4v)\varphi_0 + 8(1-v)z\varphi_1 + \omega_0\right)\right] - 2\frac{\partial \psi_0}{\partial r},$$

$$(27) \quad U_z = -(3-4v)\varphi_1 + \omega_1 = 0.$$

Based on these relations, we present the components of the stress tensor $\hat{\sigma}$ through new harmonic potentials $\varphi_0(r,\theta)$, $\varphi_1(r,\theta)$, $\omega_0(r,\theta)$, $\psi_0(r,z)$:

$$\sigma_{rr} = 0, \quad \sigma_{\theta\theta} = 0, \quad \sigma_{zz} = 0, \quad \sigma_{rz} = 0,$$

$$\sigma_{r\theta} = 2\mu\left(\frac{1}{r^2}\left((4v-3)\frac{\partial \varphi_0(r,\theta)}{\partial \theta} - 8(1-v)z\frac{\partial \varphi_1(r,\theta)}{\partial \theta}\right) - \right.$$

$$\left. -\frac{1}{r^2}\left(\frac{\partial \omega_0(r,\theta)}{\partial \theta}\right) - \left(\frac{\partial^2 \psi_0(r,z)}{\partial r^2} - \frac{1}{r}\frac{\partial \psi_0(r,z)}{\partial r}\right)\right),$$

$$(28) \quad \sigma_{\theta z} = 2\mu\left(\left(\frac{4(1-v)}{r}\frac{\partial \varphi_1(r,\theta)}{\partial \theta}\right) - \left(\frac{\partial^2 \psi_0(r,z)}{\partial r \partial z}\right)\right).$$

Thus, the boundary value problem of pure torsion of the theory of elasticity for canonical bodies is reformulated into a boundary value problem on harmonic potentials $\varphi_0(r,\theta)$, $\varphi_1(r,\theta)$, $\omega_0(r,\theta)$, $\psi_0(r,z)$:

$$\Delta\left(\varphi_0(r,\theta),\ \varphi_1(r,\theta), \omega_0(r,\theta),\ \psi_0(r,z)\right) = 0,$$

which satisfy the corresponding boundary conditions on the body surface ∂X:

$$(\vec{n}\cdot\hat{\sigma})|_{\partial X} = \left((n_\theta \sigma_{r\theta})\overrightarrow{e_r} + (n_r\sigma_{r\theta} + n_z\sigma_{\theta z})\overrightarrow{e_\theta} + (n_\theta\sigma_{\theta z})\overrightarrow{e_z}\right) =$$

$$= 2\mu\left(\left[n_r\left(\frac{1}{r^2}\left((4v-3)\frac{\partial \varphi_0(r,\theta)}{\partial \theta} - 8(1-v)\,z\frac{\partial \varphi_1(r,\theta)}{\partial \theta}\right) - \right.\right.\right.$$

$$\left.\left. -\frac{1}{r^2}\left(\frac{\partial \omega_0(r,\theta)}{\partial \theta}\right) - \left(\frac{\partial^2 \psi_0(r,z)}{\partial r^2} - \frac{1}{r}\frac{\partial \psi_0(r,z)}{\partial r}\right)\right]\overrightarrow{e_r} + \right.$$

$$+\left[n_r\left(\frac{1}{r^2}\left((4v-3)\frac{\partial \varphi_0(r,\theta)}{\partial \theta} - 8(1-v)\,z\frac{\partial \varphi_1(r,\theta)}{\partial \theta}\right) - \right.\right.$$

$$\left. -\frac{1}{r^2}\left(\frac{\partial \omega_0(r,\theta)}{\partial \theta}\right) - \left(\frac{\partial^2 \psi_0(r,z)}{\partial r^2} - \frac{1}{r}\frac{\partial \psi_0(r,z)}{\partial r}\right)\right) +$$

$$\left. +n_z\left(\left(\frac{4(1-v)}{r}\frac{\partial \varphi_1(r,\theta)}{\partial \theta}\right) - \left(\frac{\partial^2 \psi_0(r,z)}{\partial r \partial z}\right)\right)\right]\overrightarrow{e_\theta} +$$

$$\left. +\left[n_\theta\left(\left(\frac{4(1-v)}{r}\frac{\partial \varphi_1(r,\theta)}{\partial \theta}\right) - \left(\frac{\partial^2 \psi_0(r,z)}{\partial r \partial z}\right)\right)\right]\overrightarrow{e_z}\right)\Bigg|_{\partial X} = \vec{\sigma}_n^+.$$

5. Conclusion

Based on the representation of the fundamental solution of the linear elasticity theory of the mechanics of a deformable solid in the form of J.Dougall through the spatial harmonic functions $\varphi(r,\theta,z)$, $\omega(r,\theta,z)$, $\psi(r,\theta,z)$, the axisymmetric problem of the theory of elasticity in a cylindrical coordinate system for bodies bounded by a canonical surface was formulated.

The case of the pure torsion problem is considered and the elastic characteristics are given, namely, the components of deformation tensors $\hat{\varepsilon}$ and stresses $\hat{\sigma}$ and the structure of the corresponding external loads on the side surface of a given isotropic elastic body in the above-mentioned harmonic potentials. The elms on the initial basic harmonic functions in the Dougall's form are obtained, which made it possible to concretize their structure and submit through new harmonic potentials $\varphi_0(r,\theta)$, $\varphi_1(r,\theta)$, $\omega_0(r,\theta)$, $\psi_0(r,z)$ and formulate an appropriate boundary value problem.

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