Investigation streee-strain state of body with parabolic boundary

Zirakashvili Natela
I. Vekua Institute of Applied Mathematics of Iv. Javakhishvili Tbilisi State University, Georgia

natzira@yahoo.com

Abstract: *The paper stress-strain state of the homogeneous isotropic body bounded by coordinate lines of the parabolic coordinate system is studied, when on parabolic border normal or tangential stress is given. Analytical solution is obtained by the method of separation of variables. Using the MATLAB software, the numerical results are obtained of some specific problems and relevant graphs are presented.*

Keywords: *INTERNAL BOUNDARY VALUE PROBLEM, SEPARATION OF VARIABLES, PARABOLIC COORDINATES.*

1. Introduction

In the present paper the boundary value problem is considered in parabolic coordinate system ξ, η ; $-\infty < \xi < \infty,\ 0 \le \eta < \infty$ (if x, y are Cartesian coordinates, then $x = c/2(\xi^2 - \eta^2)$, $y = c\xi\eta$, where c is a scale factor and in the present paper, we take $c = 1$ [1]). Here we represent internal boundary value problem of elastic equilibrium of the homogeneous isotropic body bounded by coordinate lines of the parabolic coordinate system, when on parabolic border normal or tangential stress is given. Analytical (exact) solution is obtained using the method of separation of variables. Numerical results and corresponding graphs of above mentioned problem are presented.

2. Setting and solution problem

In domain $\Omega_1 = \{0 < \xi < \xi_1,\ 0 < \eta < \eta_1\}$ (see Fig.1), let us find a solution of the system of equilibrium equations [2]

$$\begin{aligned} &\text{a) } D_{,\xi} - K_{,\eta} = 0, \qquad \text{c) } \bar{u}_{,\xi} + \bar{\text{v}}_{,\eta} = \frac{\kappa - 2}{\kappa\mu} h_0^2 D, \\ &\text{b) } D_{,\eta} + K_{,\xi} = 0, \qquad \text{d) } \bar{\text{v}}_{,\xi} - \bar{u}_{,\eta} = \frac{1}{\mu} h_0^2 K, \end{aligned} \tag{1}$$

with respect to the unknowns D, K, u, v using the boundary conditions:

$$\text{for } \eta = \eta_1: \quad \frac{h_0^2}{2\mu}\sigma_{\eta\eta} = P,\ \frac{h_0^2}{2\mu}\tau_{\xi\eta} = 0. \tag{2}$$

$$\text{for } \eta = 0: \quad \bar{u} = 0,\quad \bar{\text{v}}_{,\eta} = 0, \tag{3}$$

$$\text{for } \xi = 0: \quad \bar{\text{v}} = 0,\quad \bar{u}_{,\xi} = 0, \tag{4}$$

$$\text{for } \xi = \xi_1: \quad u = 0,\quad \text{v} = 0, \tag{5}$$

where $\bar{u} = \dfrac{hu}{c^2}$, $\bar{\text{v}} = \dfrac{h\text{v}}{c^2}$;

$h_0 = \sqrt{\xi^2 + \eta^2}$, $h = h_\xi = h_\eta = c\sqrt{\xi^2 + \eta^2}$ are Lamé coefficients, u, v are components of the displacement vector at tangents to the coordinate lines η, ξ ; $\dfrac{\kappa - 2}{\kappa\mu} D$ is the divergence of the displacement vector, $\dfrac{K}{\mu}$ is the rotor of the displacement vector; $\sigma_{\xi\xi}$, $\sigma_{\eta\eta}$ and $\tau_{\xi\eta} = \tau_{\eta\xi}$ are normal and tangential stresses; subscripts ξ, η denote partial derivatives with respect to the corresponding coordinates; $\lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)}$, $\mu = \dfrac{E}{2(1-\nu)}$ are elastic Lamé constants; $\kappa = 4(1-\nu)$; ν is the Poisson's ratio and E is the modulus of elasticity.

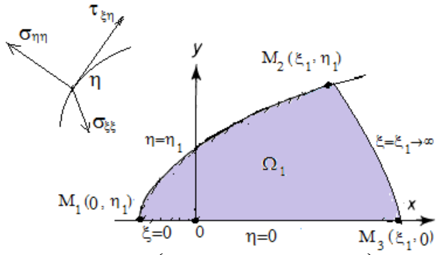


Figure 1 Area $\Omega_1 = \{0 < \xi < \xi_1,\ 0 < \eta < \eta_1\}$ bounded by parabola and line y=0.

The stress tensor components written as

$$\begin{aligned} \sigma_{\eta\eta} &= h_0^{-1}\left[\lambda\bar{u}_{,\xi} + (\lambda + 2\mu)\bar{\text{v}}_{,\eta} + \left[(\lambda+\mu) - \mu h_0^{-2}\right](\xi\bar{u} + \eta\bar{\text{v}})\right], \\ \sigma_{\xi\xi} &= h_0^{-1}\left[(\lambda + 2\mu)\bar{u}_{,\xi} + \lambda\bar{\text{v}}_{,\eta} + \left[(\lambda+\mu) + \mu h_0^{-2}\right](\xi\bar{u} + \eta\bar{\text{v}})\right], \\ \tau_{\xi\eta} &= \mu h_0^{-1}\left[(\text{v}_{,\xi} + u_{,\eta}) - h_0^{-2}(\xi\bar{\text{v}} + \eta\bar{u})\right], \end{aligned} \tag{6}$$

Boundary conditions on the linear parts $\xi = 0$ and $\eta = 0$ of consideration area enables us to continue the solutions continuously in the domain, that is the mirror reflection of the consideration area in a relationship $y = 0$ line (see Fig.2).

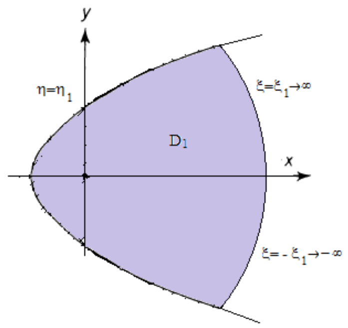


Figure 2 Area $D_1 = \{-\xi_1 < \xi < \xi_1,\ 0 < \eta < \eta_1\}$ bounded by parabola.

The solution is constructed using its general representation by two harmonic functions φ_1 , φ_2 [3].

$$\begin{aligned} \bar{u} &= -\left[\eta(\varphi_{1,\eta} - \varphi_{2,\xi}) + (\kappa - 1)\varphi_1\right]\xi + \left[\frac{\eta_1^2}{\eta}(\varphi_{1,\xi} + \varphi_{2,\eta}) \right. \\ &\quad \left. - (\kappa - 1)\varphi_2\right]\eta, \\ \bar{\text{v}} &= \left[\frac{\eta_1^2}{\eta}(\varphi_{1,\eta} - \varphi_{2,\xi}) + (\kappa - 1)\varphi_1\right]\eta + \left[\eta(\varphi_{1,\xi} + \varphi_{2,\eta}) \right. \\ &\quad \left. - (\kappa - 1)\varphi_2\right]\xi; \\ D &= \frac{\kappa\mu}{h_0^2}\left[(\varphi_{1,\eta} - \varphi_{2,\xi})\eta - (\varphi_{1,\xi} + \varphi_{2,\eta})\xi\right], \\ K &= \frac{\kappa\mu}{h_0^2}\left[(\varphi_{1,\eta} - \varphi_{2,\xi})\xi + (\varphi_{1,\xi} + \varphi_{2,\eta})\eta\right], \end{aligned} \tag{7}$$

where $\dfrac{1}{h^2}(\varphi_{i,\xi\xi} + \varphi_{i,\eta\eta}) = 0,\ \ i = 1,2.$

The stress tensor components can be written as:

$$\frac{h_0^2}{2\mu}\sigma_{\eta\eta}=-\left[\frac{\eta_1^2}{\eta}\left(\varphi_{1,\xi\xi}+\varphi_{2,\xi\eta}\right)-\frac{\kappa}{2}\varphi_{1,\eta}-\frac{\kappa-2}{2}\varphi_{2,\xi}\right]\eta$$
$$+\left[\eta\left(\varphi_{1,\xi\eta}-\varphi_{2,\eta\eta}\right)+\frac{\kappa-2}{2}\varphi_{1,\xi}-\frac{\kappa}{2}\varphi_{2,\eta}\right]\xi$$
$$-\frac{\eta_1^2-\eta}{\xi^2+\eta^2}\left[\left(\varphi_{1,\eta}-\varphi_{2,\xi}\right)\eta-\left(\varphi_{1,\xi}+\varphi_{2,\eta}\right)\xi\right],$$
$$\frac{h_0^2}{2\mu}\tau_{\xi\eta}=\left[\frac{\eta_1^2}{\eta}\left(\varphi_{1,\xi\eta}-\varphi_{2,\xi\xi}\right)+\frac{\kappa-2}{2}\varphi_{1,\xi}-\frac{\kappa}{2}\varphi_{2,\eta}\right]\eta$$
$$+\left[\eta\left(\varphi_{1,\xi\xi}+\varphi_{2,\xi\eta}\right)-\frac{\kappa}{2}\varphi_{1,\eta}-\frac{\kappa-2}{2}\varphi_{2,\xi}\right]\xi \qquad (8)$$
$$-\frac{\eta_1^2-\eta}{\xi^2+\eta^2}\left[\left(\varphi_{1,\eta}-\varphi_{2,\xi}\right)\xi+\left(\varphi_{1,\xi}+\varphi_{2,\eta}\right)\eta\right],$$

$$\frac{h_0^2}{2\mu}\sigma_{\xi\xi}=\left[\frac{\eta_1^2}{\eta}\left(\varphi_{1,\xi\xi}+\varphi_{2,\xi\eta}\right)-\frac{\kappa-4}{2}\varphi_{1,\eta}-\frac{\kappa+2}{2}\varphi_{2,\xi}\right]\eta$$
$$-\left[\eta\left(\varphi_{1,\xi\eta}-\varphi_{2,\xi\xi}\right)+\frac{\kappa+2}{2}\varphi_{1,\xi}-\frac{\kappa-4}{2}\varphi_{2,\eta}\right]\xi$$
$$+\frac{\eta_1^2-\eta}{\xi^2+\eta^2}\left[\left(\varphi_{1,\eta}-\varphi_{2,\xi}\right)\eta-\left(\varphi_{1,\xi}+\varphi_{2,\eta}\right)\xi\right],$$

The boundary conditions (3), (4) are satisfied if

$$\varphi_i=\sum_{n=1}^{\infty}\varphi_{in},\quad i=1,2, \qquad (9)$$

where

$$\varphi_{1n}=A_{1n}\sinh(n\eta)\sin(n\xi),$$
$$\varphi_{2n}=A_{2n}\cosh(n\eta)\cos(n\xi).$$

Instead of conditions (2) we have to take their equivalent following expressions

$$\frac{1}{2\mu}\left(\sigma_{\eta\eta}\cdot\eta_1-\sigma_{\xi\eta}\cdot\xi\right)=-\eta_1\left(\varphi_{1,\xi\xi}+\varphi_{2,\xi\eta}\right)-\frac{\kappa}{2}\varphi_{1,\eta}-\frac{\kappa-2}{2}\varphi_{2,\xi},$$
$$\frac{1}{2\mu}\left(\sigma_{\eta\eta}\cdot\xi+\sigma_{\xi\eta}\cdot\eta_1\right)=\eta_1\left(\varphi_{1,\xi\eta}-\varphi_{2,\xi\xi}\right)+\frac{\kappa-2}{2}\varphi_{1,\xi}-\frac{\kappa}{2}\varphi_{2,\eta}. \qquad (10)$$

3. Test problems

a. We have to solve problem (1) - (5), when $Q_1(\xi)=P$ and $Q_2(\xi)=0$, i.e. at $\eta=\eta_1$ boundary the normal load $\frac{1}{2\mu}\sigma_{\eta\eta}=\frac{P}{h_0^2}$ is given, but tangent stress is equal to zero. From (2), (9), (10) we obtain the following equations:

$$\sum_{n=1}^{\infty}\left[n^2\eta_1\sinh(n\eta_1)\left(A_{1n}+A_{2n}\right)-n\cosh(n\eta_1)\left(\frac{\kappa}{2}A_{1n}-\frac{\kappa-2}{2}A_{2n}\right)\right]\cdot\sin(n\xi)=\frac{P\eta_1}{\xi^2+\eta_1^2},$$
$$\sum_{n=1}^{\infty}\left[n^2\eta_1\cosh(n\eta_1)\left(A_{1n}+A_{2n}\right)+n\sinh(n\eta_1)\left(\frac{\kappa-2}{2}A_{1n}-\frac{\kappa}{2}A_{2n}\right)\right]\cdot\cos(n\xi)=\frac{P\xi}{\xi^2+\eta_1^2}.$$

From here obtained infinite system of the linear algebraic equations with unknown A_{1n} and A_{2n} coefficients

$$\left[\left(n^2\eta_1\sinh(n\eta_1)-n\frac{\kappa}{2}\cosh(n\eta_1)\right)A_{1n}+\left(n^2\eta_1\sinh(n\eta_1)+n\frac{\kappa-2}{2}\cosh(n\eta_1)\right)A_{2n}\right]=\widetilde{F}_{1n},$$
$$\left[\left(n^2\eta_1\cosh(n\eta_1)+n\frac{\kappa-2}{2}\sinh(n\eta_1)\right)A_{1n}+\left(n^2\eta_1\cosh(n\eta_1)-n\frac{\kappa}{2}\sinh(n\eta_1)\right)A_{2n}\right]=\widetilde{F}_{2n},\quad n=1,2,\ldots \qquad (11)$$

where $\widetilde{F}_{1n}$ and $\widetilde{F}_{2n}$ are the coefficients of expansion into Fourier series $f_1(\xi)=\sum_{n=1}^{\infty}\widetilde{F}_{1n}\sin(n\xi)$ and $f_2(\xi)=\sum_{n=1}^{\infty}\widetilde{F}_{2n}\cos(n\xi)$, respectively, $f_1(\xi)=\frac{P\eta_1}{\xi^2+\eta_1^2}$ and $f_2(\xi)=\frac{P\xi}{\xi^2+\eta_1^2}$ functions.

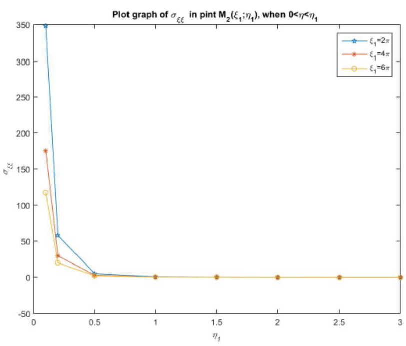
As seen, the main matrix of system (11) has a block-diagonal form, dimension of each block is 2×2 . Thus, will be solved two equations, to two A_{1n} and A_{2n} unknowns. After solving this system, we find A_{1n} and A_{2n} coefficients, next from formulas (7), (8), (9) we get displacements and stresses at any points of the body.

Numerical results obtained for some characteristic points of the body, in particular, $M_1(0,\eta_1)$, $M_2(\xi_1,\eta_1)$, $M_3(\xi_1,0)$ points (see. Fig. 1), when $0.1\le\eta_1\le 3$ for the following data: $\nu=0.3$, $E=2*10^6 kg/cm^2$, $P=-10kg/cm^2$, $0.1\le\eta_1\le 3$, $\xi_1=2*\pi$, $\xi_1=4*\pi$ and $\xi_1=6*\pi$. Numerical calculations and the visual presentation are made by MATLAB's software.

In points $M(\xi_1,\eta_1)$, ($0.1\le\eta_1\le 3$ and $\xi_1=2*\pi$, $\xi_1=4*\pi$, $\xi_1=6*\pi$) are presented graphs of values of normal $\sigma_{\eta\eta}$, tangential $\tau_{\xi\eta}$, shearing $\sigma_{\xi\xi}$ stresses, normal u and tangential v displacements, when on the parabolic boundary are given a) normal stress (see Fig.3) and b) tangential stress (see Fig.5).

In points $M_1(0,\eta_1)$, ($0.1\le\eta_1\le 3$ and $\xi_1=2*\pi$, $\xi_1=4*\pi$, $\xi_1=6*\pi$) graphs of values of tangential $\tau_{\xi\eta}$ stresses and normal u displacements, and in points $M_3(\xi_1,0)$ tangential $\tau_{\xi\eta}$ stresses and tangential v displacements are presented, when on the parabolic boundary is given a) normal stress (see Fig.4) and b) tangential stress (see Fig. 6).

The obtained results show that in case of the normal loads on the parabolic boundary the displacements are more than in case of the tangential loads.



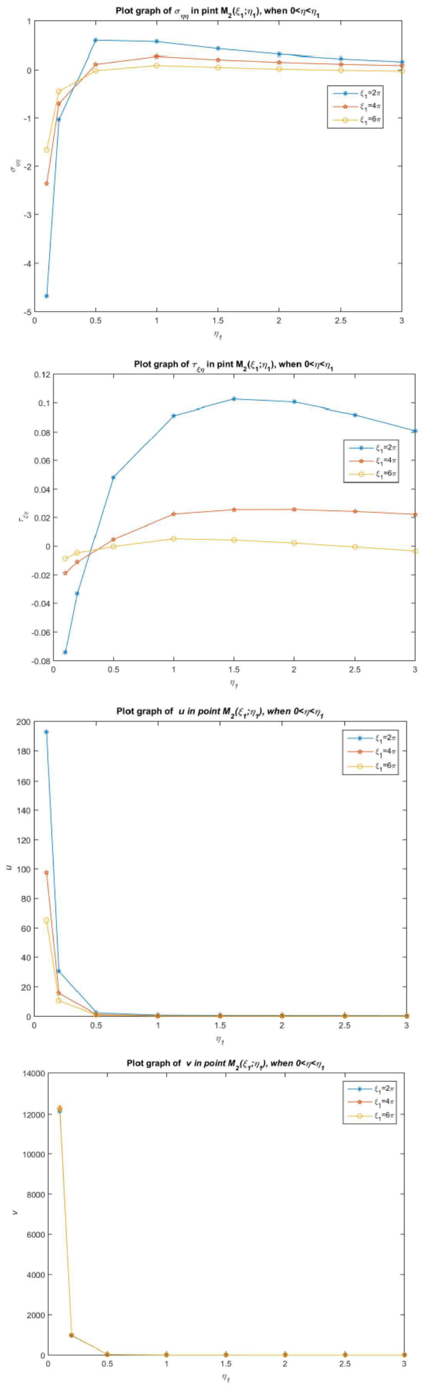
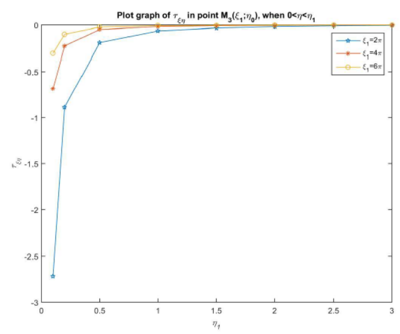


Figure 3 Stresses and displacements in points $M_2(\xi_1,\eta_1)$ for $\xi_1 = 2*\pi$, $\xi_1 = 4*\pi$ and $\xi_1 = 6*\pi$, when $0.1 \le \eta_1 \le 3$ (normal stress is applied on parabolic boundary of body).



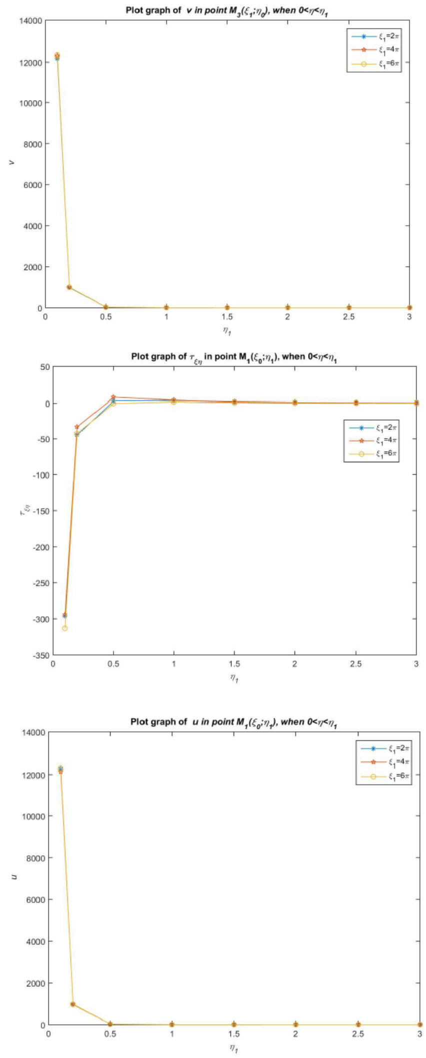
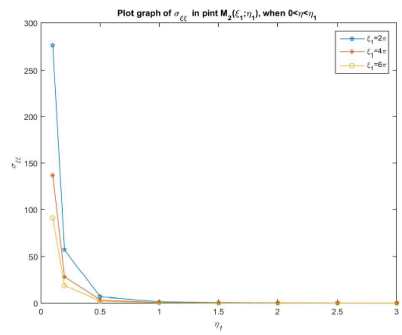


Figure 4 Tangent stress and displacements in points $M_3(\xi_1,0)$ and $M_1(0,\eta_1)$ for $\xi_1 = 2*\pi$, $\xi_1 = 4*\pi$ and $\xi_1 = 6*\pi$, when $0.1 \le \eta_1 \le 3$ (normal stress is applied on parabolic boundary of body)

b. We solve problem (1) - (5) , when $Q_1(\xi) = 0$ and $Q_2(\xi) = P$, i.e. at $\eta = \eta_1$ is given the tangent stress $\frac{1}{2\mu}\tau_{\xi\eta} = \frac{P}{h_0^2}$, but the normal stress is equal to zero.



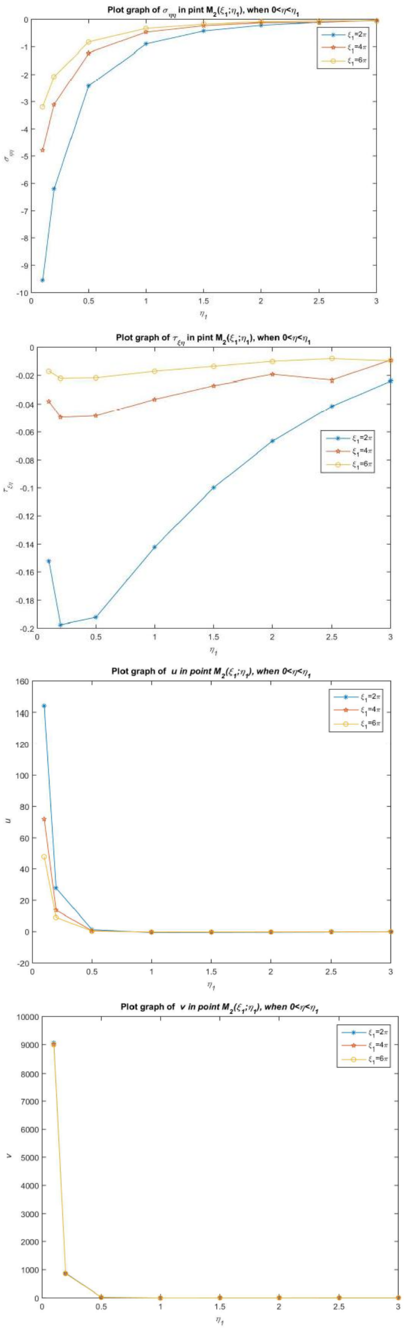
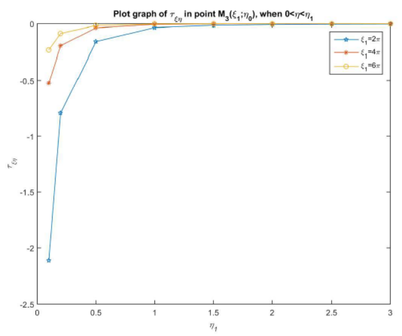


Figure 5 Stresses and displacements in points $M_2(\xi_1,\eta_1)$ for $\xi_1 = 2*\pi$, $\xi_1 = 4*\pi$ and $\xi_1 = 6*\pi$, when $0.1 \le \eta_1 \le 3$ (tangent stress is applied on parabolic boundary of body).



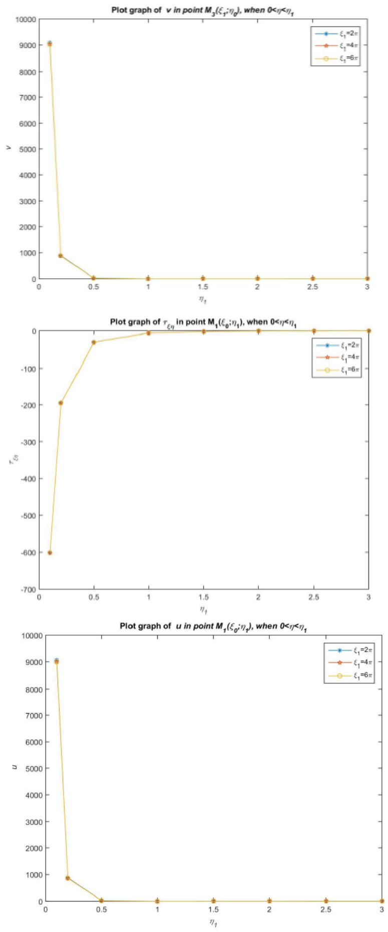


Figure 6 Tangent stress and displacements in points $M_3(\xi_1,0)$ and $M_1(0,\eta_1)$ for $\xi_1 = 2*\pi$, $\xi_1 = 4*\pi$ and $\xi_1 = 6*\pi$, when $0.1 \le \eta_1 \le 3$ (tangent stress is applied on parabolic boundary of body).

4. Conclusion

The main results of this work can be formulated as follows.

- The equilibrium equations (1) are written in terms of parabolic coordinates.
- The solution of the equilibrium equation (1) is obtained by the method of separation of variables. The solution is constructed using its general representation by two harmonic functions.
- In the parabolic coordinates exact solution of two-dimensional static boundary value problem for the elasticity is constructed for homogeneous isotropic body occupying domain bounded by coordinate lines of parabolic coordinates.

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