Generalized mathematical model of the transfer processes in the enclosing structures of buildings, constructions, thermal and engineering networks

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Abstract: *The problems of heat and moisture transfer, air permeability in single and enclosure constructions (EC) of buildings, facilities and heat, engineering and electrical networks under the influence of environmental factors and the work of heating, ventilation and air conditioning has been analyzed. A general definition of the problem taking into account the transfer processes of internal voluminous or local heat source (drainage) has been considered. A generalized mathematical model (MM) of unsteady heat and mass transfer process for bodies of different canonical form (half-plate, hollow cylinder and sphere) and their analogues has been developed. In particular cases of the mathematical model, the dependence of the physical characteristics of the (solid isotropic) medium, the boundary conditions parameters, the capacity of the mass substance transfer sources (drains) from the transfer potentials (temperature, moisture content) or the space-time continuum has been taken into consideration. The analytical solution of the generalized non-stationary and stationary heat and mass transfer problem under the general boundary conditions of different (first, second, third and mixed) kind on the outline of the researched area has been scrutinized. For constant system parameters of non-stationary transfer processes an algorithm for solving differential transfer equations using Fourier transformation with variable parameters of different kind of boundary conditions has been shown.*
For large-scale transitions, practical applications, parametric analysis of the solutions obtained, setting optimization and automation tasks for process control systems, the obtained MM, analytical and approximate solutions of direct transfer short circuits are given a criterion form convenient for these purposes.
KEY WORDS: *EQUATION, MASS AND HEAT TRANSFER, PROCESS, MATHEMATICAL MODEL, SOLUTION, CRITERIA, ENCLOSURE CONSTRUCTION, BUILDING ENVELOPE, THERMO PHYSICS OF BUILDING, FACILITIES, NETWORKS*

1. Introduction

The significance of buildings and construction facilities thermophysics is increasing nowadays due to the widespread appliance of industrial structures made of modern materials, higher demand (expectation) on the comfort of the indoor accomodations, as well as the development of civil engineering in regions of extreme climate conditions. Therefore the generation of optimal potential fields of substance and mass transfer in multilayered enclosure constructions in conjunction with automatic control systems for heat and gas supply, air conditioning and ventilation play major role in determining the project solution, the efficiency of the technical and technological processes, the mass capacity, the dimensions and the performance of the engineering equipement, the building or the entire structure as a whole.

Standard data and literary analysis of the research work [1-4] shows that computation of moist heat characterstics of exterior enclosure constructions is generally done under steady construction site parameter conditions and constant thermophysical properties of the different layers of EC. Besides that such computations and mathematical models ignore the existence of internal heat and matter sources(drainages) in the temporary periods of moisture condensation and evaporation in EC [2-5] in the periods with negative (heating period) and positive outdoor air temperatures (t ≥ 8°C). Under stationary water diffusion the humidity conditions of the EC is generally only approximately calculated applying the graphic method [1-4].Therefore one cannot stress less the relevance of the generalized formulation of the boundary problem of substance transfer process in multilayered enclosure constructions and analysis of its analytical and approximate solutions depending on the single-valuedness conditions.

2. Results and discussion

Taking into consideration conventional assumptions, air filtration and the presence of positive or negative substance transfer sources, the following generalized system of nonlinear differential equations of unsteady convective transfer in multilayered shielding constructions has been formulated:

$$\mathrm{D}\vec{\mathrm{s}}[\vec{\Pi}(\vec{\mathrm{r}},\tau)] + \mathrm{sgn}\,[\vec{\mathrm{J}}] < \nabla, \vec{\mathrm{J}}[\vec{\Pi}(\vec{\mathrm{r}},\tau)] > + \,\mathrm{sgn}[\,\vec{\mathrm{I}}]\vec{\mathrm{I}}[\vec{\Pi}(\vec{\mathrm{r}},\tau)] = \vec{0}\,, \tag{1}$$

where $\vec{s}$ - substance column-vector (scalar variables), $\vec{J}$ – stream column-vector (vector variables), $\vec{\Pi}$ – row-vector of transfer potentials or system state parameters (scalar variables), $\vec{I}$ – column-vector of positive or negative massive substance transfer sources (scalar variables), D - Lagrange derivative, $\nabla(\partial_x, \partial_y, \partial_z)$ - Hamiltonian, $\vec{r}(x, y, z)$ – vector of spatial coordinates, τ - time, $\mathrm{sgn}[\cdot]$ – the sign function, $<\cdot,\cdot>$- scalar multiplication of vectors.

The system (1) together with the equations of Navier-Stokes, continuity, thermodynamic system state parameters and the unambiguity conditions yield a closed differential equations system of the nonlinear convective transfer. Applied to fixed shielding constructions, after a number of assumption and the transformation of the non-stationary transfer potentials, the nonlinear convective transfer differential equation (1) can be presented as:

$$\vec{\Pi}_\tau(\vec{\mathrm{r}},\tau) + \mathrm{sgn}\,[\vec{\mathrm{I}}] < \nabla, \vec{\mathrm{J}}[\vec{\Pi}(\vec{\mathrm{r}},\tau)] > + \,\mathrm{sgn}[\,\vec{\mathrm{I}}]\vec{\mathrm{I}}[\vec{\Pi}(\vec{\mathrm{r}},\tau)] = \vec{0}. \tag{2}$$

To identify the only solution of the transfer equations, it is necessary to attach the initial and the BC to equation (2). We will call the functions $\vec{\Pi}(\vec{r},\tau)$ the solution of the BVP.

1. $\vec{\Pi}(\vec{r},\tau)$ defined and continuous in a closed domain $\overline{\Omega}$: $\vec{a} \le \vec{r} \le \vec{b}, \tau_0 \le \tau \le \tau_k, \tau_0 \ge 0$;

2. $\vec{\Pi}(\vec{r},\tau)$ satisfies the transport equations in the open domain Ω: $\vec{a} < \vec{r} < \vec{b}, \tau > \tau_0$;

3. $\vec{\Pi}(\vec{r},\tau)$ satisfies the initial and BC, i.e. $\vec{\Pi}(\vec{r},\tau_0) = \vec{\Pi}_0(\vec{r}),\ \ \vec{r} \in \overline{\Omega}$, $\quad l_{ij}[\vec{\Pi}] = \vec{\varphi}_{ij}(\vec{r},\tau)$, $(\vec{r},\tau) \in F_i \quad (i = \overline{1,n}, j = \overline{1,m_i}),\quad \tau \ge \tau_0$,
where $\vec{\Pi}_0(\vec{r}), \vec{\varphi}_{ij}(\vec{r},\tau)$ - are continuous functions.

4. For continuity $\vec{\Pi}(\vec{r},\tau)$ in $\overline{\Omega}$ it is necessary to satisfy the conditions of conjugation ("gluing") $\vec{\Pi}_0(\vec{r},\vec{a}) = \vec{\varphi}_{ij}(\vec{r},\tau_0) = \vec{\Pi}\ (\vec{a},\tau_0)$ and $\vec{\Pi}_0(\vec{r},\vec{b}) = \vec{\varphi}_{ij+1}(\vec{r},\tau_0) = \vec{\Pi}\ (\vec{b},\tau_0)$.

If the system parameters (thermal, thermodynamic, physicochemical and other characteristics) are constant, then the equation (2) with generalized unambiguity conditions (generalized initial and boundary conditions of the first, second, third and mixed kind) can be written in the following form:

$$\begin{cases} L[\vec{\Pi}] = \operatorname{sgn}[\vec{I}]\vec{I}\left[\vec{\Pi}(\vec{r},\tau)\right], \vec{r} \in \Omega, \tau > \tau_0, & (3) \\ L[\vec{\Pi}] = \vec{\Pi}_\tau(\vec{r},\tau) - [k]\Delta\vec{\Pi}, \\ \vec{\Pi}(\vec{r},\tau_0) = \vec{\Pi}_0(\vec{r}), \quad \vec{r} \in \overline{\Omega}, & (4) \\ l_{ij}[\vec{\Pi}] = \vec{\varphi}_{ij}(\vec{r},\tau), & (5) \\ (\vec{r},\tau) \in F_i \quad (i = \overline{1,n}, j = \overline{1,m_i}), \quad \tau \geq \tau_0, \\ l_{ij}[\vec{\Pi}] = \vec{\gamma}_{ij+1}(\vec{r},\tau) + \vec{\gamma}_{ij} < \nabla\vec{\Pi}(\vec{r},\tau), \vec{e} >, \\ F = F_1 \bigcup F_2 ... \bigcup F_n; \overline{\Omega} = \Omega \bigcup F, \\ |\vec{\gamma}_{ij}| + |\vec{\gamma}_{ij+1}| \neq 0. \end{cases}$$

Where $\vec{\Pi}$ – vector-column of transfer potentials, $[K]$ is the square matrix of constant coefficients for the interrelated transfer process or column-vector for the independent, $\vec{\Pi}_\tau(\vec{r},\tau)$ – partial derivate of transfer potentials with respect to τ, Δ – Laplacian, $\vec{\Pi}_0(\vec{r})$ – vector-function of the transfer potentials initial distribution, τ_0 – the initial time of the process, F (и F_i) – piecewise-smooth surface of the profile line (and its i part) of the range area $\overline{\Omega} \in R^n$,

R^n - n-dimensional euclidean space,

$(\vec{r},\tau) \equiv (x,y,z,\tau) \equiv (x_1,x_2,x_3,x_4) \in R^4, (x_4 \equiv \tau)$ -four-dimensional event space,

$\vec{e}$ – unit vector, $\vec{\gamma}_{ij}, \vec{\gamma}_{ij+1} = const$,

L- linear differential operator for independent variables of the second order,

l_{ij} - differential operators in $\vec{r}$ and τ of order no higher than the first (or finite ratios), and $\vec{\varphi}_{ij}$ $(\vec{r},\tau)$ - set functions, шn fact, these are the transfer potentials (temperature, moisture content, concentration, pressure, etc.) of flows flowing around the object under study that change over time.

When considering physical processes, the functions $\vec{\varphi}_{ij}$ are determined approximately from experimental data, therefore, the solution of such a mixed boundary value problem (BVP) is of practical value only if small errors in the initial and boundary conditions (BC) cannot lead to large deviations of the corresponding BVP solution. In this case, the mixed BVP is set correctly or continuously depends on the initial and BC.

Boundary problmes (3)-(5) could be solved using Fourier method [8-16]. The preliminary substitution:

$$\vec{\Pi}(\vec{r},\tau) = \vec{U}(\vec{r},\tau) + \vec{\upsilon}\left[\vec{r}, \vec{\gamma}_{ij+1}, \vec{\varphi}_{ij}(\vec{r},\tau)\right], \quad (6)$$

enables the transformation of the problem (3)-(5) relative to $\vec{\Pi}(\vec{r},\tau)$ with inhomogenous boundary conditions into a problem with homogenous boundary conditions relative to $\vec{U}(\vec{r},\tau)$:

$$\begin{cases} \left[\vec{U}(\vec{r},\tau)\right]_\tau = [K]\Delta\vec{U}(\vec{r},\tau) \pm \vec{I}_*(\vec{r},\tau), & (7) \\ \vec{U}(\vec{r},\tau_0) = \vec{U}_0(\vec{r}), & (8) \\ \vec{\gamma}_{ij} < \nabla\vec{U}, \vec{e} > + \vec{\gamma}_{ij+1}\vec{U} = \vec{0}, \vec{r} \in F_i & (9) \end{cases}$$

where $\vec{I}_*(\vec{r},\tau)$ - column-vector of the modified transfer sources.

Partial solutions of the homogenous boundary problem (7)-(9) (where $\vec{I}_*(\vec{r},\tau) = \vec{0}$) are found in a transformation with separable variables:

$$\vec{U}(\vec{r},\tau) = T(\tau)\vec{X}(\vec{r}). \quad (10)$$

Using (10) in the homogenous boundary problem we get the well known Sturm-Liouville problem for finding eigenfunctions $\vec{X}(\vec{r})$ and eigenvalues υ

$$\begin{cases} \Delta\vec{X}(\vec{r}) + \upsilon^2\vec{X}(\vec{r}) = 0, & (11) \\ \gamma_{1,i} < \nabla\vec{X}(\vec{r},\tau), \vec{e} >|_F - \gamma_{2,i}\vec{X}(\vec{r}) = 0, & (12) \end{cases}$$

The obtained coordinate functions $\vec{X}_n(\vec{r})$ are ortogonal in the region Ω for the weight $\rho(\vec{r})$ (in particular, equal: $1, r, r^2$ - accordingly, for the cylinder plate, the sphere), and $\nu_n \geq 0$ and therefore:

$$\vec{U}(\vec{r},\tau) = \sum_{n=1}^{\infty} T_n(\tau)\vec{X}_n(\vec{r}). \quad (13)$$

Furthermore expanding $\vec{I}_*(\vec{r},\tau)$ into Fourier series in $\vec{X}_n(\vec{r})$ we get:

$$\vec{I}_*(\vec{x},\tau) = \sum_{n=1}^{\infty} k_n(\tau)\vec{X}_n(\vec{r}), \quad (14)$$

where

$$k_n = \frac{1}{H_n}\int_\Omega I_*(\vec{r},\tau)\vec{X}_n(\vec{r})\rho(\vec{r})dV, \quad (15)$$

$$H_n = \int_\Omega \vec{X}_n^2(\vec{r})\rho(\vec{r})dV.$$

After all transformation, the substitution of (13) in (7) and taking into consideration (11) and (14) the transfer process equations takes the following form:

$$\sum_{n=1}^{\infty} T_n'(\tau)\vec{X}_n(\vec{r}) = -k\sum_{n=1}^{\infty} \upsilon_n^2 T_n(\tau)\vec{X}_n(\vec{r}) + \sum_{n=1}^{\infty} k_n(\tau)\vec{X}_n(\vec{r}), \quad (16)$$

and from there we have an ordinary differential equation (ODE) of first order in $T_n(\tau)$:

$$T_n'(\tau) + k\upsilon_n^2 T_n(\tau) - k_n(\tau) = 0. \quad (17)$$

Initial values of $T_n(\tau_0)$ result from (8) after we use the substitution (13) and the decomposition of $\vec{U}_0(\vec{r})$ into Fourier series in $\vec{X}_n(\vec{r})$:

$$T_n(\tau_0) = \frac{1}{H_n}\int_\Omega \vec{U}_0(\vec{r})\vec{X}_n(\vec{r})\rho(\vec{r})dV. \quad (18)$$

Now solving the ODE (17) with initial values (18) we determine the functions:

$$T_n(\tau) = e^{-k\upsilon_n^2\tau}\left[T_n(\tau_0) + \int_{\tau_0}^{\tau} k_n(t)e^{k\upsilon_n^2 t}dt\right] \quad , \qquad (19)$$

and the solution oft he boundary problem (3)-(5):

$$\vec{\Pi}(\vec{\mathrm{r}},\tau) = \sum_{n=1}^{\infty} T_n(\tau)\vec{X}_n(\vec{\mathrm{r}}) + \vec{\upsilon}[\vec{\mathrm{r}},\vec{\gamma}_{ij+1},\vec{\varphi}_{ij}(\vec{\mathrm{r}},\tau)]. \qquad (20)$$

For bodies of canonical shape (half-space, disk, hollow cylinder and sphere) and constant transfer coefficients the equation (3) is written as follows:

$$\vec{\Pi}_\tau(r,\tau) = kr^{-\Gamma}\left[r^\Gamma \vec{\Pi}_r(r,\tau)\right]_r + \mathrm{sgn}\left[\vec{I}\right]\vec{I}\left[\vec{\Pi}(r,\tau)\right] \ , \quad \tau > \tau_0 \ , \ r_0 < r < r_1 \ , \ \Gamma = \overline{0,2} \ , \qquad (21)$$

or in criteria form as:

$$\vec{P}_{Fo}(R,Fo) = R^{-\Gamma}\left[R^\Gamma \vec{P}_R(R,Fo)\right]_R + \mathrm{sgn}\left[\vec{I}_\vartheta\right]\vec{I}_\vartheta\left[\vec{P}(R,Fo)\right] \ , \quad Fo > Fo_0, \ 0 < R < 1, \ \Gamma = \overline{0,2} \ , \qquad (22)$$

Where in (21) r – current spacial coordiante, Γ – geometric quotient of the body shape ($\Gamma = 0$ for the half-space and disk, $\Gamma = 1$ for the hollow cylinder and $\Gamma = 2$ for the hollow sphere), $\vec{\Pi}_r(r,\tau)$ – derivative on the coordinate r, and in (22) - $\vec{P}, \vec{P}_{Fo}$ – the dimensionless transfer potential vector and its derivative in Fourier variable, Fo, Fo_0 - accordingly heat and mass-exchanging Fourier criteria and the Fourier criter under $\tau = \tau_0$, $R = (r - r_0)/(r_1 - r_0)$ – dimensionless coordinate, $\vec{I}_\vartheta$ – dimensionless vector of positive or negative volumetric transfer sources of substance.

A number of analytical solutions of the equation (21) for concrete Γ and their analogues under variable external(climate) parameters of boundary conditions of various kind could be found in literature [7-10 and others].

For steady-state non-linear transfer process in MEC taking into consideration air filtration and condensation and evaporation of moisture, the differential equation (21) in one of the transfer potential will have the form:

$$\left[k(\Pi)\Pi_r\right]_r + r^{-1}\Gamma k(\Pi)\Pi_r + \mathrm{sgn}\left[G\right]G(\Pi)c_p(\Pi)\Pi_r + \mathrm{sgn}\left[I\right]I(\Pi) = 0 \ , \qquad (23)$$

where $\Pi = \Pi(r) \equiv t(r)$ – temperature, G – air consumption rate, c_p – air heat capacity in constant pressure.

Instances of analytical solution of the non-linear non-homogenous transfer differential equation (23) in criteria from under different parameters are considered in [5-7].

3. Conclusions

- Generalized closed differential equation system for the process of nonlinear nonstationary convective substance transfer in multilayered enclosure constructions has been given.
- Applied to nonsteady objects (building and facility envelopes, enclosure constructions of engineering, heat or electricity networks) a mathematical model has been formalized with generalized external boundary conditions of the first, second, third and mixed kind.
- The conditions and the analytical solution algorithm has been shown for the direct nonstationary boundary problem through Fourier transformation under variable parameters for the boundary conditions of different kind.
- Boundary nonstationary and stationary nonlinear problems for bodies of cannonical shape (half-space, disk, hollow cylinder and hollow spehere) and their analogues has been stated and their possible solutions pointed out.
- The resulting mathematical models, analytical and approximate solutions oft he defined boundary trasnfer problems has been transformed into criteria form, which is conveniet for large-scale transitions, practical applications, parametric analysis oft he obtained solutions, definition of optimization problems and automation of technological process control systems.

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