ANALYTICAL ANALYSIS OF THE INVERSE STABILITY PROBLEM FOR SOME CLASSES OF NON-LINEAR DYNAMICAL SYSTEMS WITH A SMALL UNCERTAIN PARAMETER

АНАЛИЗ ОБРАТНОЙ ЗАДАЧИ УСТОЙЧИВОСТИ ДЛЯ НЕКОТОРЫХ КЛАССОВ НЕЛИНЕЙНЫХ ДИНАМИЧЕСКИХ СИСТЕМ С МАЛЫМ НЕОПРЕДЕЛЕННЫМ ПАРАМЕТРОМ

BSc Bulkina E.[1], Doctor of Technical Sciences, Professor. Firsov A.[2],
University. Institute of Computing and Control 1,2 – Peter the Great Saint-Petersburg Polytechnic University, Russia
bulkina.e.a@yandex.ru

Abstract: *In this paper we propose a method of solving inverse stability problem for some classes of non-linear dynamical systems with a small uncertain parameter. The method is based on T. Kato's perturbation theory of linear operators. We illustrate our method by solving inverse stability problem in the situation of monomerization reaction inside the cascade of chemical reactors.*
***KEYWORDS**: LOW UNCERTAINTY OF PARAMETERS, PERTURBATION THEORY, SOLUTION IN A BOUNDARY LAYER*

1. Introduction

This paper presents an analytical solution of the inverse problem of stability in the cascade process of monomerization when three chemical stirred tank reactors are used under the conditions of uncertainty of some of the reactor's parameter. The decision is based on the perturbation theory of linear operators of Kato-Rellich [1]. The process is carried out by distilling monomerization reagent from one reactor to another. The necessary cooling of the substance is carried out by the coolant in the reactor jacket. The scheme of this stage [2] is shown in Figure 1.

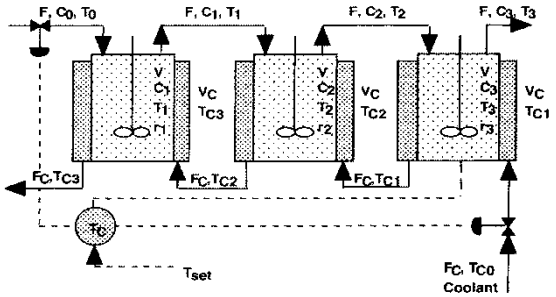


Fig.1. Scheme of the cascade of chemical reactors

The equations describing this process are as follows [2, 3, 4]:

$$\begin{aligned}
V\frac{dC_1}{dt} &= F(C_0 - C_1) - Vk(T_1)C_1,\\
\rho c_\rho V\frac{dT_1}{dt} &= F\rho c_\rho(T_0 - T_1) + Vk(T_1)C_1\Delta H - \\
&\quad -K_t A(T_1 - T_{c3}),\\
\rho_c c_{\rho c} V_c\frac{dT_{c3}}{dt} &= F_c\rho_c c_{\rho c}(T_{c2} - T_{c3}) + K_t A(T_1 - T_{c3}),\\
V\frac{dC_2}{dt} &= F(C_1 - C_2) - Vk(T_2)C_2,\\
\rho c_\rho V\frac{dT_2}{dt} &= F\rho c_\rho(T_1 - T_2) + Vk(T_2)C_2\Delta H - K_t A(T_2 - T_{c2}),\\
\rho_c c_{\rho c} V_c\frac{dT_{c2}}{dt} &= F_c\rho_c c_{\rho c}(T_{c1} - T_{c2}) + K_t A(T_2 - T_{c2}),\\
V\frac{dC_3}{dt} &= F(C_2 - C_3) - Vk(T_3)C_3,\\
\rho c_\rho V\frac{dT_3}{dt} &= F\rho c_\rho(T_2 - T_3) + Vk(T_3)C_3\Delta H - K_t A(T_3 - T_{c1}),\\
\rho_c c_{\rho c} V_c\frac{dT_{c1}}{dt} &= F_c\rho_c c_{\rho c}(T_{c0} - T_{c1}) + K_t A(T_3 - T_{c1})
\end{aligned} \quad (1)$$

The parameters of the process are given in Table 1 [2, 3, 4].

Table 1. Process parameters

№	Name	Description	Demention
1	A	Surface contact area	m^2
2	C_0	Initial concentration of the reactant in the first reactor	$kmol/m^3$
3	C_j	Concentration of the reactant in the j-th reactor	mol/m^3
4	$.\rho.$	Density of the reagent	J/kg· K
5	ρ_c	Density of the refrigerant	J/kg· K
6	E	Activation energy	J/ mol
7	F	Flow rate in the reactor	m^3 /s
8	F_c	Flow rate of the refrigerant	m^3 /s
9	R	Gas constant	J/mol· K
10	c_p	Specific heat of a substance	kg/m^3
11	c_{pc}	Specific heat of the refrigerant	kg/m^3
12	T_{cj}	Temperature of the refrigerant in the cooling part of the j-th reactor	K
33	T_{c0}	Initial temperature of the refrigerant	K
14	K_t	Coefficient of thermal conductivity	J/m^2·s·K
15	V	Volume of the reactor	m^3
16	V_c	Volume of cooling part of the reactor	m^3
17	Z	Multiplier by the Exhibitor	
18	ΔH	Thermal effect of reaction	J/ mol

From a mathematical point of view, the sustainability of monomerization is equivalent to the stability of solutions of the above system (1) of the equation. However, the technical parameters of the system can not be given exactly. The **main objective** is to evaluate the allowable errors of these parameters, which preserve the stability of the process.

2. The perturbation theory of linear operators Kato-Rellich

Let us consider a process that occurs in a dynamic system described by the following differential equation [1, 5]:

$$\dot{x} = Ax + B, \quad (2)$$

where *x*- state vector, *A* - matrix of the system parameters (matrix states), *B* - matrix of absolute terms.

We will use the method of accounting for uncertainty parameters of the system, based on the theory of perturbation of the eigenvalues of component matrices:

$$A = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots, \quad (3)$$

where we assume that $\varepsilon^2 \ll \varepsilon$. It is known that the eigenvalues of A in this case can be represented as [5]:

$$\lambda_i(\varepsilon) = \lambda_i^{(0)} + \varepsilon\lambda_i^{(1)} + \varepsilon^2\lambda_i^{(2)} + \cdots, \quad (4)$$

where $\lambda_i^{(0)}$ – is j-th eigenvalue A_0. Right and left eigenvectors corresponding to the eigenvalues $\lambda_j(\varepsilon)$, for which the following equation: $y_j^T(\varepsilon)x_j(\varepsilon) = 1$ holda true, also expandsed in power series: $x_j(\varepsilon) = x_j^{(0)} + \varepsilon x_j^{(1)} + \varepsilon^2 x_j^{(2)} + \cdots$, $y_j(\varepsilon) = y_j^{(0)} + \varepsilon y_j^{(1)} + \varepsilon^2 y_j^{(2)} + \cdots$. And eigenvalues $\lambda_j^{(1)}$ и $\lambda_j^{(1)}$ are calculated as follows:

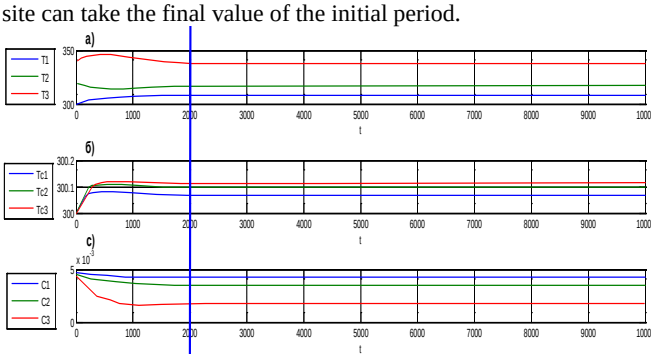
Step 1. Calculate the eigenvalues λ_j of matrix A_0.
Step 2: Finding the right and the left normalized eigenvectors A_0: $(A_0 - \lambda_j I)x_j = 0, y_j^T(A_0 - \lambda_j I) = 0, |x_j| = 1, |y_j| = 1$.
Step 3. The matrix Z_{k0} is constructed: $Z_{k0} = \frac{\prod_{j=1,j\neq k}^{s}(A_0 - \lambda_j I)}{\prod_{k=1,j\neq k}^{s}(\lambda_k - \lambda_j)}$, where s is dimention of A.
Step 4. The matrix E_j is calculated: $E_j = \sum_{k=1,k\neq n}^{s} \frac{1}{\lambda_j - \lambda_k} Z_{k0}$.
Step 5. Calculated coefficients of the expansion (4) by the formulas: $\lambda_j^{(1)} = y_1^T A_1 x_1$, $\lambda_j^{(2)} = tr(BE_1A_2Z_{10})$.
In this task we assume that $\varepsilon^3 \ll \varepsilon^2$. Therefore, in the above formula (4) it is sufficient to consider the first three terms.
As a criterion for the stability of the system we will use the root criterion, that is, the permissible range of the parameter ε, *which* is characterized by a system of inequalities of the form for sufficiently small α:

$$\lambda_j(\varepsilon) < -\alpha^2, \quad (5)$$

Solution of the inequality (5) is the intersection of the intervals that are solutions of each inequality. It is obvious that the intersection is not empty, due to the fact since the solution of each inequality will contain zero.

3. *Reaching the main objective*

Perturbation operators theory developed by Kato, refers to linear operators in function spaces, and the system of equations (1) is non-linear. However, the system (1) can be linearized. Indeed, the experiment [2,3] (Figure 1) shows that the dynamics of changes in the main characteristics of the system admits a linearization, from some point of time from the beginning of the process, and this initial period of time is small relative to the total process time. Thus, during the initial period of time, as it is so small, it is possible to seek the solution of the system by expanding in the time series, and the remaining slot to linearize in the usual manner. At the same time as the initial value of the interval being we linearized site can take the final value of the initial period.



1. a) The plot of a change of a temperature; b) the plot of the change of the temperature of the coolant; c) the graph of the concentration change of the substance

3.1. *Constructing an analytical solution of the system at the initial time interval*

To unify the system (1) we introduce the following notation: $C_i = X_i$, $T_i = X_{i+3}$, $T_{ci} = X_{i+6}$, $i = 1,2,3$. Then the system of equations for the first reactor takes the following form (the other equations will be similar):

$$\frac{dX_1}{dt} = \frac{F(C_0 - X_1)}{V} - Zexp\left(\frac{-E}{RX_4}\right)X_1,$$

$$\frac{dX_4}{dt} = \frac{F(T_0 - X_4)}{V} + \frac{ZX_1\Delta H}{\rho c_\rho} e^{\left(\frac{-E}{RX_4}\right)} - \frac{K_t A(X_4 - X_9)}{\rho c_\rho V}, \quad (7)$$

$$\frac{dX_9}{dt} = \frac{F(X_8 - X_9)}{V_c} + \frac{K_t A(X_4 - X_9)}{\rho_c c_{\rho c} V_c}.$$

Denote $t = T\tau$, where T – the time course of the whole process, and $X_j = \bar{X}_j x_j$, $j = 1,2,..,9$. Note, that x_j и τ – dimensionless quantities and $\tau \ll 1$. Then the system describing the dynamics of the process in the first reactor takes the form (the other equations will take the same form):

$$\frac{dx_1}{d\tau} = \frac{FTC_0}{V\bar{X}_1} - \frac{FT}{V}x_1 - ZTe^{\left(\frac{-E}{R\bar{X}_4 x_4}\right)}x_1,$$

$$\frac{dx_4}{d\tau} = \frac{FTT_0}{V\bar{X}_4} - \frac{FT}{V}x_4 + \frac{ZT\bar{X}_1 x_1 \Delta H}{\rho c_\rho \bar{X}_4} e^{\left(\frac{-E}{R\bar{X}_4 x_4}\right)} + \frac{K_t AT(x_4 - \frac{\bar{X}_9}{\bar{X}_4}x_9)}{\rho c_\rho V}$$

$$\frac{dx_9}{d\tau} = \frac{F_c T(\frac{T_{c2}}{\bar{X}_9} - x_9)}{V_c} + \frac{K_t AT(\frac{\bar{X}_4}{\bar{X}_9}x_4 - x_9)}{\rho_c c_{\rho c} V_c}.$$

We seek a solution of the system on a small initial period of time as a power series in the small parameter τ: $x_j(\tau) = C_0^j + C_1^j\tau + C_2^j\tau^2 + \cdots$, where C_0^j are known and the initial values of parameters of the input vector. The expression $e^{\left(\frac{-E}{R\bar{X}_4 x_4}\right)}$ is expanded in a series as a function of the τ: $e^{\left(\frac{-E}{R\bar{X}_4 x_4}\right)} = e^{\left(\frac{-E}{R\bar{X}_4 C_0^4}\right)} + \frac{e^{\left(\frac{-E}{R\bar{X}_4 C_0^4}\right)} EC_1^4}{R\bar{X}_4 (C_0^4)^2}\tau + \frac{1}{2}\frac{e^{\left(\frac{-E}{R\bar{X}_4 C_0^4}\right)} E(2R\bar{X}_4 (C_0^4)^2 C_2^4 - 2R\bar{X}_4 C_0^4 (C_1^4)^2 + E(C_1^4)^2)}{(R\bar{X}_4 (C_0^4)^2)^2}\tau^2 + o(\tau^3)$.

Representation of the solution of the system in the form of a power series in the variable τ and its further substitution in the system of equations gives, by equating the coefficients of corresponding powers of τ, a recurrent system of equations for the coefficients of the expansion of the solution. Here is a small piece of this scheme:

$$C_1^1 = \frac{FTC_0}{V\bar{X}_1} - \frac{FTC_0^1}{V} - ZTe^{\left(\frac{-E}{R\bar{X}_4 C_0^4}\right)}C_0^1,$$

$$C_1^4 = \frac{FTC_0}{V\bar{X}_4} - \frac{FTC_0^4}{V} - \frac{ZT\Delta H}{\rho c_\rho} e^{\left(\frac{-E}{R\bar{X}_4 C_0^4}\right)} C_0^1 - \frac{K_t AT\left(C_0^4 - C_0^9\frac{\bar{X}_9}{\bar{X}_4}\right)}{\rho c_\rho V},$$

$$C_2^1 = \frac{1}{2}\left(-ZTe^{\left(\frac{-E}{R\bar{X}_4 C_0^4}\right)}C_1^1 - \frac{ZTe^{\left(\frac{-E}{R\bar{X}_4 C_0^4}\right)} EC_1^4 C_0^1}{R\bar{X}_4 (C_0^4)^2} - \frac{FTC_1^1}{V}\right).$$

So it's clear that all the coefficients are calculated consistently.

3.2. *Linearization of the system of differential equations of the dynamics of chemical reactors*

In the previous section solution of the system at the initial time interval was constructedl. Now we construct a solution of (1) in the remaining time interval by its linearization. Note that the solution obtained in the above initial time interval at the end of this interval is the vector of initial values for the remaining time interval.
Represent x_j as a follows $x_j = X_j^0(1 + y_j)$, $j = 1,2..9$, where y_j – small quantity, which allows to neglect the terms containing the second order y_j. The main issue arises when converting linearization terms of the form $e^{\frac{c}{x_j}}$. For this we use the Taylor

formula:

$$e^{-\frac{c}{x_j}} = e^{-\frac{c}{x_j^{(0)}(1+y_j)}} = e^{-\frac{c}{x_j^{(0)}}} + e^{-\frac{c}{x_j^{(0)}}}\left(\frac{c}{x_j^{(0)}(1+y_j)}\right)'_{x_j=0} \cdot x_j + O(x_j^2).$$

Neglecting the second order x_j^2, we obtain approximately: $e^{-\frac{c}{x_j}} = e^{-\frac{c}{x_j^{(0)}(1+y_j)}} = e^{-\frac{c}{x_j^{(0)}}} + e^{-\frac{c}{x_j^{(0)}}} \cdot \frac{c}{x_j^{(0)}} \cdot x_j$. Thus, the system of equations for the first reactor takes the form:

$$\begin{aligned} \frac{dy_1}{d\tau} &= \xi_{11}y_1 + \xi_{14}y_4 + b_1 \\ \frac{dy_4}{d\tau} &= \xi_{41}y_1 + \xi_{44}y_4 + \xi_{49}y_9 + b_4, \\ \frac{dy_9}{d\tau} &= \xi_{94}y_4 + \xi_{99}y_9 + b_9, \end{aligned} \qquad (8)$$

where $\xi_{11} = \left(-\beta e^{-\frac{c}{x_4^{(0)}}} - \alpha\right), X_1^{(0)}, \xi_{14} = \beta \frac{X_1^{(0)}}{X_4^{(0)}} e^{-\frac{c}{x_4^{(0)}}}, \xi_{41} = \frac{\beta_H X_1^{(0)}}{\delta} e^{-\frac{c}{x_4^{(0)}}}, \xi_{44} = X_4^{(0)}\left(-\alpha - \frac{\gamma}{\varphi}\right) + \beta_H c \frac{X_1^{(0)}}{X_4^{(0)}} e^{-\frac{c}{x_4^{(0)}}}, \xi_{94} = \frac{\gamma\rho}{\varphi_c} X_4^{(0)}, \xi_{99} = X_9^{(0)}\left(-\alpha_c - \frac{\gamma}{\varphi_c}\right),$ but $b_1 = \left(-\beta e^{-\frac{c}{x_4^{(0)}}} - \alpha\right) X_1^{(0)}, b_2 = \left(-\alpha - \frac{\gamma}{\varphi}\right) X_4^{(0)} + \frac{\gamma}{\varphi\eta} X_9^{(0)} - X_1^{(0)} \frac{\beta_H}{\delta} e^{-\frac{c}{x_4^{(0)}}} - \alpha\theta_4, b_3 = \left(X_9^{(0)} + \theta_9\right)\alpha_c + \frac{\gamma\eta}{\varphi_c} X_4^{(0)}.$ Where $\varphi = \delta V, \varphi_c = \delta_c V_c, \beta_H = \beta\Delta H, \eta = \frac{X_4^{(0)}}{X_9^{(0)}}, \theta_4 = \frac{T_0}{X_4^{(0)}}, \theta_9 = \frac{T_c}{X_9^{(0)}},$ где $\alpha = \frac{FT}{V}, \beta = ZT, \gamma = K_t AT, \alpha_c = \frac{F_c T}{V_c}, \delta = \rho c_\rho, \delta_c = \rho_c c_{\rho c}.$

So we obtain:

$$\mathrm{A} = \begin{pmatrix} \xi_{11} & \xi_{14} & 0 \\ \xi_{41} & \xi_{44} & \xi_{49} \\ 0 & \xi_{94} & \xi_{99} \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, x = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

4. *Uncertainty parameters of the matrix A*

Let's suppose that the uncertainty in parameters reflecting the heat of reaction -ΔH.

Suppose, therefore, that $\Delta H^0 = \Delta H(1+\varepsilon), |\varepsilon| \ll 1$. Then $\beta_H^0 = \beta\Delta H(1+\varepsilon)$, and, in turn, the matrix parameters ξ_{41} and ξ_{44} take the form:

$$\xi_{41} = \frac{\beta_H^0 X_1^{(0)}}{\delta} e^{-\frac{c}{x_4^{(0)}}} = \frac{\beta\,\Delta H^0 X_1^{(0)}}{\delta} e^{-\frac{c}{x_4^{(0)}}} = \frac{\beta\,\Delta H}{\delta} e^{-\frac{c}{x_4^{(0)}}}(1+\varepsilon)X_1^{(0)} = \frac{\beta\,\Delta H}{\delta} e^{-\frac{c}{x_4^{(0)}}} X_1^{(0)} + \left(\frac{\beta\,\Delta H}{\delta} e^{-\frac{c}{x_4^{(0)}}} X_1^{(0)}\right)\varepsilon = \xi_{41}^0 + \xi_{41}^1,$$

$$\xi_{44} = X_4^{(0)}\left(-\alpha - \frac{\gamma}{\varphi}\right) + \beta_H^0 c \frac{X_1^{(0)}}{X_4^{(0)}} e^{-\frac{c}{x_4^{(0)}}} = X_4^{(0)}\left(-\alpha - \frac{\gamma}{\varphi}\right) + \beta\,\Delta H^0 c \frac{X_1^{(0)}}{X_4^{(0)}} e^{-\frac{c}{x_4^{(0)}}} = X_4^{(0)}\left(-\alpha - \frac{\gamma}{\varphi}\right) + \beta\,\Delta H c \frac{X_1^{(0)}}{X_4^{(0)}} e^{-\frac{c}{x_4^{(0)}}}(1+\varepsilon) = X_4^{(0)}\left(-\alpha - \frac{\gamma}{\varphi}\right) + \beta\,\Delta H c \frac{X_1^{(0)}}{X_4^{(0)}} e^{-\frac{c}{x_4^{(0)}}} + \left(\beta\,\Delta H c \frac{X_1^{(0)}}{X_4^{(0)}} e^{-\frac{c}{x_4^{(0)}}}\right)\varepsilon = \xi_{44}^0 + \xi_{44}^1.$$

Thus, we obtain:

$$\mathrm{A} = A_0 + \varepsilon A_0 = \begin{pmatrix} \xi_{11} & \xi_{14} & 0 \\ \xi_{41}^0 & \xi_{44}^0 & \xi_{49} \\ 0 & \xi_{94} & \xi_{99} \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ \xi_{41}^1 & \xi_{44}^1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

5. *Summary*

In this paper, we propose a method of analysis of a nonlinear dynamical system, which describes by a standard process of monomerization cascade of chemical substances in the stirred tank reactors, provided that certain parameters of the reactor have undefined values. Namely: 1) an algorithm to obtain an approximate analytical solution of the corresponding system of nonlinear differential equations has been developed; 2) on the basis of the solution a fairly simple algorithm for its numerical implementation has been derived; 3) the corresponding system of differential equations is transformed to a form, which allows you to explicitly take into account the uncertainty of the parameters of the system, and, therefore, to use the methods of perturbation theory of linear operators by Rellich - Kato.

6. *Литература*

1) T. Kato, Perturbation theory for linear operators. - M .: Mir, 1972, 740 p. (in Russian)

2) Ingham J, Dunn I, Heinzle E, Prenosil E, Shape B. Chemical Engineering Dynamics – WILEY-VCH Verlag GmbH & Co.KGaA, 1977. 643 p.

3) V.A. Kholodnov. System analysis and decision making. Mathematical modeling and optimization of chemical engineering objects / V.A. Kholodnov, A. C.. Gumerov, N.N. Valeev, V.M. Emelyanov, V.N. Chepikova, M.Y. Lebedev // Tutorial. SPb .: SPbSTI (TU), 2006. 340 p. (in Russian)

4) V. M/ Krylov. Theory and practice of mathematical modeling. / V. M. Krylov, V. A. Kholodnov // Tutorial. SPb .: SPbSTI (TU), 2007. 178

5) Lancaster P, Tismenetsky M. The Theory of Matrices Second Edition. - New York: Academic Press, 1985, 570 p.

6) A.N. Firsov, S.L. Chulin. Constructing an analytical solution of a system of nonlinear differential equations describing the dynamic processes in chemical reactors. - System analysis in the ingenering and control. Proceedings of the XVI International scientific-practical conference. Conference. Part 1. SPb .: Publishing House of the Polytechnic. University Press, 2012, 145-151 p. (in Russian)

7) A.N. Firsov, E.A. Bulkina. Constructing of analytical solution of the system of nonlinear differential equations describing the initial process of monomerization in the cascade of chemical reactors. // System analysis in engineering and management: Proceedings of the XVIII International scientific and practical conference: best papers of the conference. – SPb.: Publishing of Polytechnic University, 2015. - P. 150-155.