ANALYTICAL ANALYSIS OF THE INVERSE STABILITY PROBLEM FOR SOME CLASSES OF NON-LINEAR DYNAMICAL SYSTEMS WITH A SMALL UNCERTAIN PARAMETER

Abstract: In this paper we propose a method of solving inverse stability problem for some classes of non-linear dynamical systems with a small uncertain parameter. The method is based on T. Kato’s perturbation theory of linear operators. We illustrate our method by solving inverse stability problem in the situation of monomerization reaction inside the cascade of chemical reactors.

KEYWORDS: LOW UNCERTAINTY OF PARAMETERS, PERTURBATION THEORY, SOLUTION IN A BOUNDARY LAYER

1. Introduction

This paper presents an analytical solution of the inverse problem of stability in the cascade process of monomerization when three chemical stirred tank reactors are used under the conditions of uncertainty of some of the reactor’s parameter. The decision is based on the perturbation theory of linear operators of Kato-Rellich

\[ F \approx E + \frac{1}{2} (F - E^2) \]

From a mathematical point of view, the sustainability of monomerization is equivalent to the stability of solutions of the above system (1) of the equation. However, the technical parameters of the system can not be given exactly. The main objective is to evaluate the allowable errors of these parameters, which preserve the stability of the process.

2. The perturbation theory of linear operators Kato-Rellich

Let us consider a process that occurs in a dynamic system described by the following differential equation [1, 5]:

\[ \dot{x} = Ax + B, \]

where \( x \) - state vector, \( A \) - matrix of the system parameters (matrix states), \( B \) - matrix of absolute terms.
We will use the method of accounting for uncertainty parameters of the system, based on the theory of perturbation of the eigenvalues of component matrices:

\[ A = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \ldots, \]

where we assume that \( \varepsilon^2 \ll \varepsilon \). It is known that the eigenvalues of \( A \) in this case can be represented as [5]:

\[ \lambda_j(\varepsilon) = \lambda_j^{(0)} + \varepsilon \lambda_j^{(1)} + \varepsilon^2 \lambda_j^{(2)} + \ldots, \]

where \( \lambda_j^{(0)} \) is \( j \)-th eigenvalue \( A_0 \). Right and left eigenvectors corresponding to the eigenvalues \( \lambda_j(\varepsilon) \), for which the following equation:

\[ y_j(\varepsilon)x(\varepsilon) = 1 \]

holds true, also expanded in power series:

\[ y_j(\varepsilon) = y_j^{(0)} + \varepsilon y_j^{(1)} + \varepsilon^2 y_j^{(2)} + \ldots, \]

and eigenvalues \( \lambda_j^{(1)} \) and \( \lambda_j^{(2)} \) are calculated as follows:

**3. Reaching the main objective**

Perturbation operators theory developed by Kato, refers to linear operators in function spaces, and the system of equations (1) is non-linear. However, the system (1) can be linearized. Indeed, the experiment [2,3] (Figure 1) shows that the dynamics of changes in the main characteristics of the system admits a linearization, from some point of time from the beginning of the process, and this initial period of time is small relative to the total process time. Thus, during the initial period of time, as it is so small, it is possible to seek the solution of the system by expanding in the time series, and the remaining slot to linearize in the usual manner. At the same time as the initial value of the interval being linearized site can take the final value of the initial period.

1. a) The plot of a change of a temperature; b) the plot of the change of the temperature of the coolant; c) the graph of the concentration change of the substance

3.1. Constructing an analytical solution of the system at the initial time interval

To unify the system (1) we introduce the following notation:

\[ C_i = X_i, \quad T_i = X_{i+3}, \quad T_{ci} = X_{i+6}, \quad i = 1,2,3. \]

Then the system of equations for the first reactor takes the following form (the other equations will be similar):

\[ \frac{dX_1}{dt} = \frac{F(C_0 - X_1)}{V} - \frac{Z_T}{\rho c} \frac{1}{\rho c} C_1, \]

\[ \frac{dX_2}{dt} = \frac{F(T_0 - X_2)}{V} + \frac{Z_T}{\rho c} \frac{1}{\rho c} C_1, \]

Denote \( t = T \), where \( T \) – the time course of the whole process, and \( X_1 = X_j \), \( j = 1,2,3 \). Note, that \( X_j \) and \( t \) – dimensionless quantities and \( t \ll 1 \). Then the system describing the dynamics of the process in the first reactor takes the form (the other equations will take the same form):

\[ \frac{dX_1}{dt} = \frac{FTC_0}{VX_1} - \frac{FT}{VX_1} \frac{T_0}{T} X_1 \frac{Z_T}{\rho c} \frac{1}{\rho c} C_1, \]

\[ \frac{dX_2}{dt} = \frac{FTC_0}{VX_1} - \frac{FT}{VX_1} \frac{T_0}{T} X_1 \frac{Z_T}{\rho c} \frac{1}{\rho c} C_1, \]

We seek a solution of the system on a small initial period of time as a power series in the small parameter \( \varepsilon \) \( x_j(\varepsilon) = C_j^0 + C_j^1 \varepsilon + C_j^2 \varepsilon^2 + \ldots \), where \( C_j^0 \) are known and the initial values of parameters of the input vector. The expression \( \varepsilon(\varepsilon^2 x_j + e_{\varepsilon^2} C_1) \) is expanded in a series as a function of the \( \varepsilon \):

\[ \varepsilon(\varepsilon^2 x_j + e_{\varepsilon^2} C_1) = \varepsilon e_{\varepsilon^2} C_1 + e_{\varepsilon^2} C_1 C_1 + \varepsilon^2 e_{\varepsilon^2} C_1 (2C_1^2 + C_1 e_{\varepsilon^2} C_1) + o(\varepsilon^2). \]

3.2. Linearization of the system of differential equations of the dynamics of chemical reactors

In the previous section solution of the system at the initial time interval was constructed. Now we construct a solution of (1) in the remaining time interval by its linearization. Note that the solution obtained in the above initial time interval at the end of this interval is the vector of initial values for the remaining time interval. Represent \( x_j \) as a follows \( x_j = \xi_j^0 + y_j, \quad j = 1,2,3 \), where \( y_j \) – small quantity, which allows to neglect the terms containing the second order \( y_j \). The main issue arises when converting linearization terms of the form \( e^\xi \). For this we use the Taylor
formula:
\[ e^{-\frac{c}{x_j^2}} = e^{-\frac{c}{x_j^{(1+\delta x_j^2)}}} = e^{-\frac{c}{x_j^2}} + \frac{c}{x_j^2} \left( \frac{c}{x_j^2} \right)^{1+\delta x_j^2} x_j + O(x_j^2). \]

Neglecting the second order term, we obtain approximately: \( e^{-\frac{c}{x_j^2}} \approx \frac{c}{x_j^2} \). Thus, the system of equations for the first reactor takes the form:
\[
\begin{align*}
\frac{d\xi_{11}}{dt} &= \xi_{11} Y_1 + \xi_{14} Y_4 + b_1, \\
\frac{d\xi_{44}}{dt} &= \xi_{41} Y_1 + \xi_{44} Y_4 + b_4, \\
\frac{d\xi_{44}}{dt} &= \xi_{99} Y_4 + \xi_{99} Y_9 + b_9,
\end{align*}
\]
where \( \xi_{11} = \left( -\beta \cdot \frac{x_4}{x_4^2} - \alpha \right), \xi_{14} = \beta \cdot \frac{x_4}{x_4^2} \).

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Thus, we obtain: \( \frac{d\xi_{11}}{dt} = \xi_{11} Y_1 + \xi_{14} Y_4 + b_1 \)

5. Summary

In this paper, we propose a method of analysis of a nonlinear dynamical system, which describes by a standard process of monomerization cascade of chemical substances in the stirred tank reactors, provided that certain parameters of the reactor have undefined values. Namely: 1) an algorithm to obtain an approximate analytical solution of the corresponding system of nonlinear differential equations has been developed; 2) on the basis of the solution a fairly simple algorithm for its numerical implementation has been derived; 3) the corresponding system of differential equations is transformed to a form, which allows you to explicitly take into account the uncertainty of the parameters of the system, and, therefore, to use the methods of perturbation theory of linear operators by Rellich - Kato.

6. Литература