«MOMENT» REPRESENTATION OF «FAST DECREASING» GENERALIZED FUNCTIONS AND THEIR APPLICATION TO SEVERAL APPLIED STOCHASTIC PROBLEMS

“МОМЕНТНОЕ” ПРЕДСТАВЛЕНИЕ “БЫСТРО УБЫВАЮЩИХ” ОБОБЩЁННЫХ ФУНКЦИЙ И ИХ ПРИЛОЖЕНИЕ К НЕКОТОРЫМ ПРИКЛАДНЫМ СТОХАСТИЧЕСКИМ ЗАДАЧАМ

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Abstract: *This paper describes the process of building special space of generalized functions, its properties and applications. Presented applications are: constructive solution of Kolmogorov-Feller type equation with polynomial drift coefficient; proof of exponential nature of equilibrium establishment in rarefied gas, described by Boltzmann equation of kinetic theory of gases.*

Keywords: GENERALIZED FUNCTIONS, MOMENT REPRESENTATION, KOLMOGOROV-FELLER EQUATION, BOLTZMANN EQUATION

1. Introduction

This paper describes the construction, properties and application of special space of generalized functions (see terminology in [1]), proposed by A.N. Firsov in [2, 3] and developed in cooperation with A.V. Koval [4]. The following examples of application are presented: the constructive solution of the Kolmogorov-Feller type equation with the polynomial drift coefficient, and the proof of the exponential nature of the time equilibrium establishment in a rarified gas, described by the Boltzmann kinetic equation.

Kolmogorov-Feller type equations are found in control theory, communication theory and stellar dynamics. In literature on analytical methods for solving such equations the case of drift coefficient linear dependence on coordinate is usually considered. In this case it is possible to use the Fourier transformation. This paper considers the solution of the Kolmogorov-Feller equation with quadratic and cubic drift coefficient – the cases, in which the use of the Fourier transformation is ineffective.

In addition in this paper the asymptotic properties (for the large values of time) of the linearized Boltzmann equation solution are considered.

Boltzmann kinetic equation is the basis of research and analysis of mathematical models of (mass, energy, momentum) transport processes in gases, as derivation of the equation is based on the more natural idea of gas as a set of interacting molecules, rather than a continuous medium. This approach is particularly advantageous in the research of the body motion in the upper atmosphere, where the idea of gas as a continuous medium is inadequate. The main problem, that arises here, is the complexity of the Boltzmann equation, which consists in the structure of the collision integral operator, associated with the microstructure of the medium and the nature of the interaction between the matter molecules. Therein lies one of the reasons for the comparatively small number of studies on the analytically rigorous formulation of the solvability conditions and the analysis of the solutions of the Boltzmann equation. The latter refers not only to the full nonlinear version of the Boltzmann equation, but its linear approximation.

2. The theory of “fast decreasing” generalized functions [2,3]

Definition 1. Let $s > 0$. Through E_s we denote the space of (complex) functions $\varphi \in C^\infty(R^\nu)$ such, that for any $\rho > 0$

$$|D^q\varphi(x)| \le C(s+\rho)^{|q|}e^{(s+\rho)|x|}, x \in R^\nu.$$

Here C is a constant, depending, generally speaking, on φ, s and ρ, but not on $q \in \mathbb{Z}_0^\nu$.

Note, that the polynomials $P(x_1, x_2, \dots, x_\nu)$ and functions $e^{s_1x_1+\dots+s_\nu x_\nu}$, $e^{i(s_1x_1+\dots+s_\nu x_\nu)}$ belong to the E_s, the latter at $s \ge max(|s_1|, \dots, |s_\nu|)$.

We introduce a countable system of norms in E_s

$$\|\varphi\|_s^{(\rho)} = \sup_{q,x}\left[\frac{|D^q\varphi(x)|}{(s+\rho)^{|q|}}e^{-(s+\rho)|x|}\right], \rho = 1, \frac{1}{2}, \frac{1}{3}, \dots \quad (1)$$

Theorem 1. Space E_s, equipped with the system of norms (1), is a complete countably-normed space.

Since $E_{s+1} \supset E_s$ and convergence of the sequence $\{\varphi_n\}$ in E_{s+1} results from its convergence in E_s, it is possible to introduce a countable union $E = \bigcup_{s=1}^\infty E_s$; convergence in E is determined in a conventional manner (see [3]). Obviously, space E is complete in terms of the corresponding convergence. The following property of spaces E_s is the basis for further derivations.

Theorem 2. Let $\varphi \in E_s$, $a = (a_1, a_2, \dots, a_\nu) \in R^\nu$. Then:

1) Taylor series for the φ

$$\varphi(x) = \sum_{l=0}^{\infty}\sum_{|q|=l}\frac{\varphi^{(q)}(a)}{q!}(x-a)^q$$

converges for all $x \in R^\nu$;

2) partial sums

$$S_m(x) = \sum_{l=0}^{m}\sum_{|q|=l}\frac{\varphi^{(q)}(a)}{q!}(x-a)^q$$

converge to φ in the terms of convergence in E_s.

The following lemmas indicate a number of other properties of space E (details see in. [2, 3]).

Lemma 1. If $\varphi, \psi \in E$, then the product $\varphi\psi \in E$.

Lemma 2. Let $\varphi_n \to \varphi$, $\psi_n \to \psi$ in space E, then $\varphi_n\psi_n \to \varphi\psi$ in E.

Lemma 3. If $\varphi \in E_s$, then $D^q\varphi \in E_s$.

Lemma 4. If $\varphi_n \to \varphi$ in E_s, then $D^q\varphi_n \to D^q\varphi$ in E_s.

Space E' is introduced in conventional manner as a conjugate to E. In the usual manner it determines linear operations, the operation of multiplication by functions of E and differentiation. These operations, as follows from the previous results, are continuous in terms of convergence in E' (i.e. weak convergence). By the theorem on completeness of space conjugated to complete countably normed space, this space will be complete (with respect to weak convergence) – see [1].

It should be noted here, that the stock of the regular functionals in E' is large enough. So every summable in R^νfunction $f(x)$, satisfying the condition

$$f(x) = O\left(e^{-\alpha|x|^{1+\varepsilon}}\right), |x| \to \infty, \alpha > 0, \varepsilon > 0,$$

generates in E' the functional $\hat{f}$ by the formula

$$(\hat{f}, \varphi) = \int_{R^\nu} f(x)\varphi(x)dx, \varphi \in E.$$

The following lemma takes place:

Lemma 5. If $f \in L_1(R^\nu)$, $\varphi \in E$ and for every $\int_{R^\nu} f(x)\varphi(x)dx = 0$, then $f(x) = 0$ almost everywhere.

Also we note, that if the “normal” function $f(x)$ is differentiable in the usual sense, and f and $f^{(q)}$ generate the regular functionals $\hat{f}$ and $\widehat{f^{(q)}}$, then $\widehat{f^{(q)}} = \hat{f}^{(q)}$, where the right is the functional $\hat{f}$ derivative in terms of differentiation in space E'. Finally, the delta-function $\delta_a = \delta(x-a)$, defined in the usual way, i.e. $(\delta_a, \varphi) = \varphi(a)$, $\varphi \in E$, is also belong to E' and is a singular functional.

Now we formulate the main properties of the generalized functions of E'(details – in [2,3]):

Theorem 3.

1) Every generalized function $f \in E'$ at a random point $a \in \mathbb{R}^n$ can be presented in the form

$$f = \sum_{k=0}^{\infty} \sum_{|q|=k} C_a^{(q)} \delta_a^{(q)},$$

where coefficients $C_a^{(q)} = (-1)^{|q|} \frac{(f,(x-a)^q)}{q!}$.

2) The generalized function f belongs to E' if and only if the series

$$\sum_{k=0}^{\infty} s^k \sum_{|q|=k} \left| C_a^{(q)} \right| \tag{2}$$

converges for every $s > 0$.

3) Let $f, g \in E'$ и $C_a^{(q)}$, $d_a^{(q)}$ be "moment" decomposition coefficients. Then the following statements are valid:

1. $\alpha f + \beta g = \sum_{k=0}^{\infty} \sum_{|q|=k} \left(\alpha C_a^{(q)} + \beta d_a^{(q)} \right) \delta_a^{(q)}$, where $\alpha, \beta \in \mathbb{C}$;
2. $D^k f = \sum_{k=0}^{\infty} \sum_{|q|=k} C_a^{(q)} \delta_a^{(q)}$;
3. if $\psi \in E$, then $\psi f = \sum_{k=0}^{\infty} \sum_{|q|=k} h_a^{(q)} \delta_a^{(q)}$, where

$$h_a^{(q)} = \sum_{r=0}^{\infty} (-1)^r \sum_{\substack{|n|=r+|q| \\ n \geq q}} \binom{n}{n-q} C_a^{(q)} \psi^{(q)}(a).$$

In the space E' it is possible to use the usual procedure to determine the convolution of two generalized functions. Such convolution always exists (unlike other spaces of generalized functions), has the usual properties and satisfies the following

Theorem 4. Let $f, g \in E'$ and $C_a^{(q)}$, $d_a^{(q)}$ are corresponding "moment" decomposition coefficients. Then

$$f * g = \sum_{k=0}^{\infty} \sum_{|q|=k} h_a^{(q)} \delta_a^{(q)}, \text{ where}$$

$$h_a^{(q)} = \sum_{i+j \leq q} (-1)^{|q-i-j|} \frac{a^{q-i-j}}{(q-i-j)!} C_a^{(i)} d_a^{(j)}.$$

In particular, for $a = 0$ "moment" decomposition coefficients

$$h_0^{(q)} = \sum_{i+j=q} C_0^{(i)} d_0^{(j)}.$$

3. Solution of the Kolmogorov-Feller equation with the quadratic drift coefficient [4]

One-dimensional stationary Kolmogorov-Feller equation with the quadratic drift coefficient is given by:

$$\frac{d}{dx}[\alpha(x)W(x)] + \nu \int_{-\infty}^{+\infty} p(A)W(x-A)dA - \nu W(x) = 0. \tag{3}$$

Let the unknown function $W(x)$ and the given function $p(A)$ be the generalized functions of E'. Then, according to the first paragraph of theorem 3, function $W(x)$ can be presented as $W(x) = \sum_{q=0}^{\infty} C_0^{(q)} \delta_0^{(q)}$. Similarly $p(A) = \sum_{q=0}^{\infty} d_0^{(q)} \delta_0^{(q)}$, where $d_0^{(q)} = (-1)^q \frac{(p,A^q)}{q!}$. Since the meaning of the function $p(A)$ is probability density, we should assume that $(p,1) = 1$.

We consider separately each of the terms in the left side of equation (3). With obvious transformations, using the statements of the third paragraph of theorem 3, we obtain

$$\frac{d}{dx}[(\alpha x + \beta x^2)W(x)] = \sum_{q=0}^{\infty} k_0^{(q)} \delta_0^{(q)} + \sum_{q=0}^{\infty} l_0^{(q)} \delta_0^{(q)},$$

where $k_0^{(q)} = \alpha C_0^{(q)} - 2\beta(q+1)C_0^{(q)}$,

$$l_0^{(q)} = \beta(q+1)(q+2)C_0^{(q)} - \alpha(q+1)C_0^{(q)}.$$

According to the theorem 2 we transform the convolution in the equation (3) to the form

$$p * W = \sum_{q=0}^{\infty} \left(\sum_{i=0}^{q} d_0^{(q)} C_0^{(q)} \right) \delta_0^{(q)}.$$

Thus, equation (3) reduces to

$$\sum_{q=0}^{\infty} k_0^{(q)} \delta_0^{(q)} + \sum_{q=0}^{\infty} l_0^{(q)} \delta_0^{(q)} + \nu \sum_{q=0}^{\infty} \left(\sum_{i=0}^{q} d_0^{(q-i)} C_0^{(i)} \right) \delta_0^{(q)} - \nu \sum_{q=0}^{\infty} C_0^{(q)} \delta_0^{(q)} = 0,$$

which is equivalent to

$$\sum_{q=0}^{\infty} \left[k_0^{(q)} + l_0^{(q)} + \nu \sum_{i=0}^{q} d_0^{(q-i)} C_0^{(i)} - \nu C_0^{(q)} \right] \delta_0^{(q)} = 0.$$

Using paragraph 1 of theorem 3, we find, that

$$k_0^{(q)} + l_0^{(q)} + \nu \sum_{i=0}^{q} d_0^{(q-i)} C_0^{(i)} - \nu C_0^{(q)} = 0 \;\; \forall q \in \mathbb{Z}_0,.$$

As a result, after appropriate transformations, taking into account the property $(p,1) = 1$, we obtain a recurrence relation, from which one can sequentially find all the coefficients $C_0^{(q)}$:

$$C_0^{(q+1)} = \frac{\alpha}{\beta(q+1)} C_0^{(q)} - \frac{\nu}{\beta q(q+1)} \sum_{i=0}^{q-1} d_0^{(q-i)} C_0^{(i)}. \tag{4}$$

To obtain a unique solution of the chain of equations (4), apparently, it is necessary to set the coefficients $C_0^{(0)}$ and $C_0^{(1)}$. Thus, to settle the issue of the existence of the equation (3) solution in the space E', one must verify the point 2 of theorem 3.

Theorem 5. To satisfy point 2 of theorem 3 it suffices that the sequence $\left\{ C_0^{(q)} \right\}_{q=0}^{+\infty}$ of (4) is bounded.

The proof of theorem 5 is based on the following lemma.

Lemma 6. Suppose that there exists $Q > 0$ such that for any $0 \leq q \leq Q$ the requirement $\left| C_0^{(q)} \right| \leq \left| C_0^{(Q)} \right|$ is fulfilled. Suppose, moreover, that $\left| \frac{\alpha}{\beta} \right| + \left| \frac{\nu}{\beta} \right| D \leq Q + 1$, where $D = \max_{q \geq 0} \left| d_0^{(q)} \right|$. In this case $\left| C_0^{(q)} \right| \leq \left| C_0^{(Q)} \right|$ for any $q \geq Q$.

To prove this lemma the principle of mathematical induction is used.

Here is an example solution of equation (3) with (4). Suppose $p(A)$ is a regular functional, which is a normal distribution with the mean 0 and variance σ. In this case "moment" decomposition coefficients are

$$d_0^{(q)} = (-1)^q \frac{(p,A^q)}{q!} = \begin{cases} 0, q = 2i+1, i \in \mathbb{Z}_0, \\ \frac{\sigma^q}{i!!}, q = 2i, i \in \mathbb{Z}_0. \end{cases}$$

Then the moments of the unknown function $W(x)$ can be found, using the relation

$$C_0^{(q+1)} = \frac{\alpha}{\beta(q+1)} C_0^{(q)} - \frac{\nu}{\beta q(q+1)} \sum_{i=0}^{q-1} \frac{\sigma^q}{2 \cdot 4 \cdot \ldots \cdot q} C_0^{(i)},$$
$$q = 2j, \;\; j \in \mathbb{Z}_0$$

4. Solution of the Kolmogorov-Feller equation with the cubic drift coefficient

One-dimensional stationary Kolmogorov-Feller equation with the cubic drift coefficient has the form

$$\frac{d}{dx}[(\alpha x + \beta x^2 + \gamma x^3)W(x)] + \nu \int_{-\infty}^{+\infty} p(A)W(x-A)dA - \nu W(x) = 0. \tag{5}$$

The chain of equations (4) is replaced with relations:

$$C_0^{(q+2)} = -\frac{\alpha}{\gamma(q+1)(q+2)} C_0^{(q)} + \frac{\beta}{\gamma(q+2)} C_0^{(q+1)} + \frac{\nu}{\gamma q(q+1)(q+2)} \sum_{i=0}^{q-1} d_0^{(q-i)} C_0^{(i)} \tag{6}$$

There is a theorem analogous to theorem 5:

Theorem 6. Suppose the sequence of coefficients (6) $\left\{ C_0^{(q)} \right\}_{q=0}^{+\infty}$ is bounded, then the condition 2 of theorem 3 is satisfied.

The proof of this theorem is based on a lemma analogous to lemma 6.

5. The analysis of the asymptotic behavior of the linearized Boltzmann equation solutions for large values of time [3]

Consider the Cauchy problem for the linearized Boltzmann kinetic equation [5]:

$$\begin{gathered} \frac{\partial f}{\partial t} + \bar{u}\frac{\partial f}{\partial \bar{x}} = L[f], L[f] = K[f] - \nu f \\ f = f(\bar{x},\bar{u},t), \bar{x} \in \mathbb{R}^3, \bar{u} \in \mathbb{R}^3, t \geq 0 \\ f|_{t=0} = f_0(\bar{x},\bar{u}) \end{gathered} \tag{7}$$

Here $f(\bar{x},\bar{u},t)$ is linearized distribution function of the molecules in the coordinates $\bar{x}$ and velocities $\bar{u}$ at time t. $K[f]$ is a bounded linear operator, acting on f as a function $\bar{u}$; $\nu = \nu(u) = O(u^\beta)$ for $u \longrightarrow \infty$, $0 < \beta \leq 1$, $u = |\bar{u}|$. Properties of the function $\nu(u)$ depend on the specific intermolecular interaction model, accepted during the derivation of the kinetic equations. For details see [5].

It is known [6] that the solution of (7) in the case of "hard" intermolecular interaction potentials $U = C_k\, r^{-k}$, $k > 5$ at $t \to \infty$ has, in general, the power asymptotics of the form $O\left(\frac{1}{1+t^\mu}\right)$, $\mu > 0$. This result is obtained under assumption that $f(\bar{x},\bar{u},t)$ at $x = |\bar{x}| \to \infty$ behaves as a function of the $L_p(\mathbb{R}_x^3), p > 1$.

It turns out, that if we impose more rigid conditions on $f(\bar{x},\bar{u},t)$ behavior, for example, require that $f(\bar{x},\bar{u},t)$ satisfies for $\bar{x}$ (and uniformly for $\bar{u}, t$):

$f(\bar{x},\bar{u},t) = O\left(exp(-\alpha|\bar{x}|^{1+\varepsilon})\right), |\bar{x}| \to \infty, \alpha > 0, \varepsilon > 0,$

then the establishment of equilibrium (i.e. function f convergence to zero at $t \longrightarrow \infty$) is exponentially fast.

The idea of the proof is as follows. We seek a $f(\bar{x},\bar{u},t)$ in the class of functions such that almost for every $\bar{u} \in \mathbb{R}^3$ and every > 0 $f(\bar{x},\bar{u},t) \in E'_x$, i.e. the function f can be written as

$$f(\bar{x},\bar{u},t) = \sum_{l=0}^{\infty} \sum_{|q|=l} C_a^{(q)}(\bar{u},t)\, \delta_a^{(q)}(\bar{x}). \tag{8}$$

Substituting this expression in (7) and taking into account the results of paragraph 2 of this paper, we obtain the infinite "looping" system of equations for coefficients $C_a^{(q)}(\bar{u},t)$

$$\begin{aligned} &\frac{\partial C_a^{(0)}}{\partial t} = L\left[C_a^{(0)}\right] \\ &\vdots \\ &\frac{\partial C_a^{(q)}}{\partial t} = L\left[C_a^{(q)}\right] - \left[u_1 C_a^{(q-I_1)} + u_2 C_a^{(q-I_2)} + u_3 C_a^{(q-I_3)}\right], \\ &|q| \neq 0 \end{aligned} \tag{9}$$

where in terms of I_1, I_2, I_3 multi-indices (1, 0, 0), (0, 1, 0) and (0, 0, 1) are expressed respectively.

Equations (9) are heterogeneous equations of the form

$$\frac{\partial C_a^{(q)}}{\partial t} = L\left[C_a^{(q)}\right] - g_q(\bar{u},t), |q| \neq 0,$$

where $g_q(\bar{u},t)$ is a known function (at each step – its own known function). Thus, the properties of the functions $C_a^{(q)}(\bar{u},t)$ depend on the operator L properties. The latter have been studied sufficiently (see [3,5-9]). In particular, on the subspace of functions $w(\bar{u},t)$, orthogonal in terms of $L_2(\mathbb{R}_u^3)$ to the subspace of additive invariants (which is actually equivalent to the classical conservation laws for gas), the operator L generates a semigroup $T(t), t > 0$ of bounded operators, which solves an abstract Cauchy problem for equation (7); in addition, it turns out, that $\|T(t)\| \leq const \cdot e^{-\mu t}, \mu > 0$. With the method, similar to the one used in [6,7], by induction we obtain an estimate for the solutions of equations (9) of the form (the norm is meant in terms of $L_2(\mathbb{R}_u^3)$): $\left\|C_a^{(q)}(t)\right\| \leq const \cdot e^{-\gamma t}, \gamma > 0$, where $const$ depends on the initial distribution function $f_0(\bar{x},\bar{u})$ and parameters of the operator L.

The latter estimate, taking into account (8), leads to the conclusion of an exponentially fast (in time) equilibrium establishment in the system, described by the problem (7).

6. Results

This paper describes the construction, properties and applications of a special space of the generalized functions, proposed by A.N. Firsov [2,3] and developed jointly with A.V. Koval [4]. The following applications are presented: a constructive solution of the Kolmogorov-Feller equation with the quadratic drift coefficient and a proof of the exponential in time nature of equilibrium establishment in rarified gas, described with the kinetic Boltzmann equation.

The proposed method of "moment" representation of generalized functions can be effectively used for solving differential equations with polynomial coefficients and convolution equations.

7. References

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