NUMERICALLY-ANALYTICAL SOLUTION OF THE TRANSPORTATION PROBLEM FOR THE VISCOUS WEAKLY COMPRESSIBLE LIQUID, MOVING THROUGH THE PIPELINE WITH NON-STATIONARY BOUNDARY CONDITIONS

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ЧИСЛЕННО-АНАЛИТИЧЕСКОЕ РЕШЕНИЕ ЗАДАЧИ О ТРАНСПОРТИРОВКЕ ВЯЗКОЙ СЛАБОСЖИМАЕМОЙ ЖИДКОСТИ ПО ТРУБОПРОВОДУ ПРИ НЕСТАЦИОНАРНЫХ КРАЕВЫХ УСЛОВИЯХ

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Abstract: *In the paper we solve the problem of transporting viscous weakly compressible liquid through the pipeline of the circular cross-section under non-stationary conditions. This paper is based on previous author's results, presented on WS of XIII MTM Congress. The Navier-Stokes equations are the basis for mathematical model. The liquid kinematic viscosity and its density are considered to be weakly changing with time. The non-stationarity is caused by specific boundary conditions, depending on time. The obtained results allow to optimize the control of a viscous weakly compressible liquid flows in the pipeline systems.*

Keywords: *WEAKLY COMPRESSIBLE LIQUID, PIPELINE, NON-STATIONARY HYDRODYNAMICS, NAVIER-STOKES EQUATIONS*

1. Introduction

Physical and chemical properties of the fluids, transported through pipeline systems, define the project and operating parameters of pipelines in many ways. Considering oil industry, pipelines are used for the transportation of a wide spectrum of hydrocarbons and its mixtures, that greatly differing in chemical and physical properties: products of oil refining (gasoline, jet fuel, kerosene, diesel fuel, fuel oil, etc.), petrochemical raw materials (benzene, styrene, propylene, etc.), aromatic hydrocarbons (xylene, toluene, cumene, etc.), liquefied petroleum fuel (liquefied natural gas, liquefied petroleum gas) and so on.

Information about fluid physical properties is considered both when choosing the type of the mathematical model, and when defining coefficients (parameters) of corresponding elements of this model. Changing, these parameters may have influence on the flow character.

Previously we have studied the effect of the variable dynamic viscosity, depending on a small parameter, on the liquid velocity in the pipeline [1]. In this paper we consider the case, when the liquid density is not constant; for example, hydrocarbons, generally, are weakly compressible liquid, which density depends on pressure and temperature [2]. We assume, that the density is changing with time according to the known manner. Also, we consider non-stationary conditions at the pipe edges (in terms of variable pressure).

2. Formulation of the problem

The task is to obtain the equation of a viscous weakly compressible liquid motion through the cylindrical pipe with non-stationary boundary conditions. We take the system of equations of a viscous liquid dynamics as a basis [3]:

$$\begin{cases} \dfrac{d\rho}{dt} + \rho div\vec{\upsilon} = 0, \\ \rho\dfrac{d\vec{\upsilon}}{dt} = \rho\vec{F} + \dfrac{\partial\vec{\tau}_x}{\partial x} + \dfrac{\partial\vec{\tau}_y}{\partial y} + \dfrac{\partial\vec{\tau}_z}{\partial z}, \end{cases} \tag{1}$$

– continuity equation and motion equation. Here $\vec{\upsilon}$ – velocity, ρ – density, $\vec{F}$ – mass forces vector, $\vec{\tau}_x$, $\vec{\tau}_y$, $\vec{\tau}_z$ – vectors, corresponding to the stress tensor lines [3].

Consider the flow along the horizontal pipe axis (z axis), neglecting mass forces ($\vec{F} = 0$). Based on these assumptions, rewrite the system (1) as follows:

$$\begin{cases} \dfrac{d\rho}{dt} + \rho\dfrac{\partial\upsilon_z}{\partial z} = 0, \\ \rho\left(\dfrac{\partial\vec{\upsilon}}{\partial t} + \upsilon_z\dfrac{\partial\vec{\upsilon}}{\partial z}\right) = \dfrac{\partial\vec{\tau}_x}{\partial x} + \dfrac{\partial\vec{\tau}_y}{\partial y} + \dfrac{\partial\vec{\tau}_z}{\partial z}. \end{cases} \tag{1'}$$

Project the second equation of the system ($1'$) on the axes of the corresponding Cartesian coordinate system:

$$\rho\left(\frac{\partial\upsilon_x}{\partial t} + \upsilon_z\frac{\partial\upsilon_x}{\partial z}\right) = \frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z};$$

$$\rho\left(\frac{\partial\upsilon_y}{\partial t} + \upsilon_z\frac{\partial\upsilon_y}{\partial z}\right) = \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\tau_{yy}}{\partial y} + \frac{\partial\tau_{zy}}{\partial z};$$

$$\rho\left(\frac{\partial\upsilon_z}{\partial t} + \upsilon_z\frac{\partial\upsilon_z}{\partial z}\right) = \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\tau_{zz}}{\partial z}.$$

Taking into account the assumptions, made above, we obtain the following expressions:

$$(\lambda + \mu)\frac{\partial^2\upsilon_z}{\partial z\partial x} = \frac{\partial p}{\partial x},$$

$$(\lambda + \mu)\frac{\partial^2\upsilon_z}{\partial y\partial z} = \frac{\partial p}{\partial y},$$

$$\rho\left(\frac{\partial\upsilon_z}{\partial t} + \upsilon_z\frac{\partial\upsilon_z}{\partial z}\right) = \mu\left(\frac{\partial^2\upsilon_z}{\partial x^2} + \frac{\partial^2\upsilon_z}{\partial y^2} + \frac{\partial^2\upsilon_z}{\partial z^2}\right) - \frac{\partial p}{\partial z} + \lambda\frac{\partial^2\upsilon_z}{\partial z^2},$$

where $p = p(t,z)$ – pressure, λ – the bulk viscosity coefficient, μ – the dynamic viscosity coefficient. The first two equations yield the following:

$$\begin{cases} (\lambda + \mu)\dfrac{\partial\upsilon_z}{\partial z} = p + F_1(y,z,t), \\ (\lambda + \mu)\dfrac{\partial\upsilon_z}{\partial z} = p + F_2(x,z,t). \end{cases}$$

Hence $F_1(y,z,t) = F_2(x,z,t) = F(z,t)$ and, as a result:

$$p = (\lambda + \mu)\frac{\partial \upsilon_z}{\partial z} - F(z,t). \qquad (2)$$

From the continuity equation the velocity z derivative can be expressed:

$$\frac{\partial \upsilon_z}{\partial z} = -\frac{1}{\rho}\frac{d\rho}{dt}; \qquad (3)$$

then

$$\rho\frac{\partial \upsilon_z}{\partial t} - \upsilon_z\frac{d\rho}{dt} = \mu\left(\frac{\partial^2\upsilon_z}{\partial x^2} + \frac{\partial^2\upsilon_z}{\partial y^2} + \frac{\partial^2\upsilon_z}{\partial z^2}\right) - \frac{\partial p}{\partial z} + \lambda\frac{\partial^2\upsilon_z}{\partial z^2}. \qquad (4)$$

Since we noted at the beginning, that the liquid density is changing only with time, then $\frac{\partial^2\upsilon_z}{\partial z^2} = 0$ (according to equation (3)).

From (2) we obtain for $\frac{\partial p}{\partial z}$:

$$\frac{\partial p}{\partial z} = -\frac{\partial F(z,t)}{\partial z}.$$

Hence the pressure variation along the pipeline length is described by some function $f(t,z) = -\frac{\partial F(z,t)}{\partial z}$.

Turning to the cylindrical coordinate system in equation (4) and taking into account the last obtained relations, we write the equation for determining the viscous compressible liquid velocity in the pipeline:

$$\frac{\partial \upsilon_z}{\partial t} = \nu(t)\frac{\partial^2\upsilon_z}{\partial r^2} + \frac{\nu(t)}{r}\frac{\partial \upsilon_z}{\partial r} + \frac{1}{\rho(t)}\frac{d\rho(t)}{dt}\upsilon_z + \frac{1}{\rho(t)}f(z,t), \qquad (5)$$

where $t \in [0,T]$, $r \in [0,R]$. For the equation (5) we formulate the initial-boundary problem: find the $\upsilon_z(t,r,z)$ function, which is a solution of the equation (5), at the initial time turns into the given function $\upsilon_z(0,r,z) = \varphi(r,z)$ (in other words, the initial distribution), and satisfies the boundary condition $\upsilon_z(t,R,z) = 0$ at all times. If we need to provide predetermined velocity values $\upsilon_z(t,r,0)$ and $\upsilon_z(t,r,L)$ at the pipe inlet and outlet respectively, then the $f(t,z)$ function will be determined, based on these requirements, during the solution process. However, for simplicity, in this paper we will assume, that the function $f(t,z)$ is given, and, hence, we may omit the conditions at the pipe edges.

3. The solution

The equation (5) is a non-homogeneous heat equation with variable coefficients. If we reduce all the relations to dimensionless form, we will obtain the following equation:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial u}{\partial \eta} + a(\tau)u + \gamma(\tau,\chi), \qquad (5')$$

where $\tau \in [0,\overline{T}]$, $\eta \in [0,N]$, $a(\tau) = \frac{1}{\rho(\tau)}\frac{d\rho}{d\tau}$. This equation is more convenient to solve, because there is only one variable coefficient.

We assume that the density is weakly changing with time. We divide the time period $[0,\overline{T}]$ in h small intervals, so that we may consider the density and hence the $a(\tau)$ coefficient to be constant over each interval. Then we get h equations of the form ($5'$), the solution of each of which has the form:

$$u^i(\tau,\eta,\chi) = $$
$$= e^{a_i\tau}\left[\int_0^N G_1(\tau,\xi,\eta)\psi^i(\xi,\chi)d\xi + \int_{\tau^i_{min}}^{\tau}\int_0^N G_2(\tau - s,\xi,\eta)\gamma^i(s,\chi)d\xi ds\right], \qquad (6)$$

where

$$G_1(\tau,\xi,\eta) = \sum_{n=1}^{\infty}\frac{2\xi}{N^2 J_1^2(\mu_n)}e^{-\frac{\mu_n^2}{N^2}\left(\tau - \tau^i_{min}\right)}J_0\left(\frac{\mu_n}{N}\xi\right)J_0\left(\frac{\mu_n}{N}\eta\right),$$

$$G_2(\tau - s,\xi,\eta) = \sum_{n=1}^{\infty}\frac{2\xi}{N^2 J_1^2(\mu_n)}e^{-\frac{\mu_n^2}{N^2}(\tau - s)}J_0\left(\frac{\mu_n}{N}\xi\right)J_0\left(\frac{\mu_n}{N}\eta\right),$$

$$\psi^i(\xi,\chi) = \frac{d}{\nu}e^{-a_i\tau^i_{min}}\varphi^i(\xi,\chi),$$

$$\gamma^i(s,\chi) = \frac{d^3}{\nu^2}\frac{f^i(s,\chi)}{\rho_m},$$

$$\tau \in \left[\tau^i_{min},\tau^i_{max}\right), \; i = \overline{1,h}.$$

As an example, we build the solution in case, when velocity doesn't depend on z. We take the Poiseuille velocity distribution as an initial condition [3], set the linear law of density variation, and linear change in pressure at the pipe outlet:

$$\rho(t) = \frac{a}{T}t + \rho_0, \qquad (7)$$

$$p(t,z) = \left(-\frac{z}{L} + 1\right)p_H + \frac{z}{L}\left(\frac{p_k - p_0}{T}t + p_0\right). \qquad (8)$$

We pay special attention to the oil transportation, and oil density may vary within $730 - 940$ $\frac{kg}{m^3}$ [5]; assume $\rho_0 = 830.$ $\frac{kg}{m^3}$ as an initial value of the density.

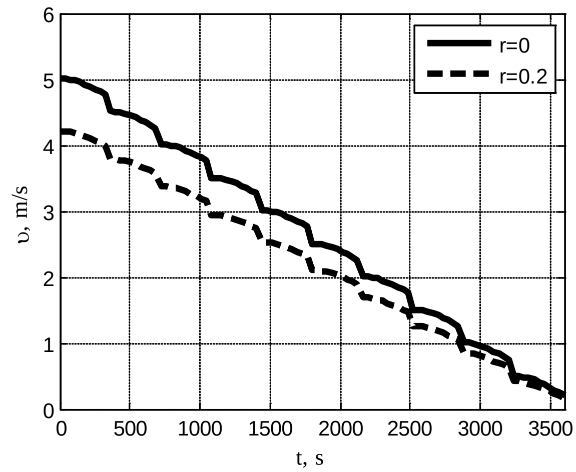

Fig. 1. *Liquid velocity dependence on time at different distances from the pipe axis*

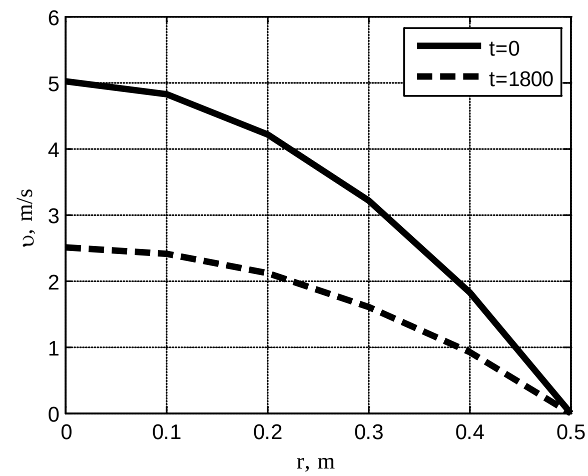

Fig. 2. *Liquid velocity dependence on radial coordinate at different moments of time*

It is interesting to see, how much effect does the variable density have on flow velocity. To do this we plot $\upsilon_z(t,0)$, one for the $\rho = 830 \ \frac{kg}{m^3}$, and the second one for the density, varying according to the equation (7) (fig.3).

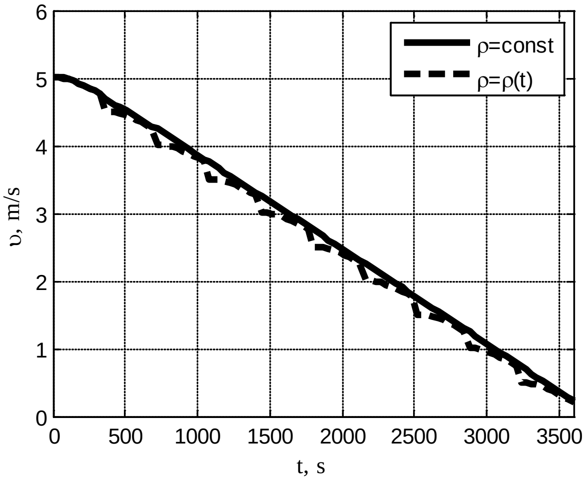

Fig. 3. *Flow velocity dependence on time at the pipe axis*

4. Conclusion

In this paper the problem of a viscous weakly compressible liquid motion through the horizontal pipe of circular cross-section with non-stationary boundary conditions is solved. The density is weakly changing in time, and non-stationarity is dictated by the boundary conditions at the ends of the pipeline. The numerically-analytical solution, described with equation (6), was obtained. Comparison of the solution obtained with a solution for the case of constant density (fig.3) allows us to say, that if the liquid density is weakly changing in time, it has no significant effect on the motion velocity.

These results will allow, in perspective, to solve the problem of optimizing the control of viscous weakly compressible liquids flows through the pipeline systems.

5. References

[1] Sorokina N. The solution of the problem of one-parameter perturbation of the viscous incompressible liquid motion through straight round pipe/ Sorokina N// Machines. Technologies. Materials. – 2016. – Issue 4/2016. – P.20-22

[2] Davidson V.E. Fundamentals of fluid dynamics in examples^ Tutorial for high school students. – M.: Izdat.. centr. «Akademiya», 2008. – 320 p.

[3] Vallander, S.V. Lectures on hydroaeromechanics. – L.: Izd-vo Leningr. un-ta, 1973, 296 p. (in Russian)

[4] Sorokina N., Firsov A. The problem of viscous incompressible liquid motion through the cylindrical round pipe with non-stationary boundary conditions. In. III international scientific and technical conference “Technics. Technologies. Education. Safety”: Proceedings, Vol.2. Veliko Tarnovo, Bulgaria, 28-29 May 2015, P.9-10 (in Russian)

[5] Physical quantities: Handbook / I.S. Grigoriev, E.Z. Meilikh. – M.: Energoatomizdat, 1991. – 1232 p. (in Russian)