NUMERICAL-ANALYTICAL METHOD FOR SOLVING THE INVERSE PROBLEM OF STABILITY FOR TECHNICAL SYSTEMS WITH MULTIPLE UNCERTAIN PARAMETERS

ЧИСЛЕННО-АНАЛИТИЧЕСКИЙ МЕТОД РЕШЕНИЯ ОБРАТНОЙ ЗАДАЧИ УСТОЙЧИВОСТИ ДЛЯ ТЕХНИЧЕСКИХ СИСТЕМ С НЕСКОЛЬКИМИ НЕОПРЕДЕЛЕННЫМИ ПАРАМЕТРАМИ

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Abstract: *The paper considers the problem of determining the boundaries of possible changes of parameters of dynamic system whilst preserving stability of the system. The proposed method for determining such bounders is based on the research on the perturbation theory of matrix eigenvalues which depend on several perturbation parameters (by Ji-guang Sun). The results of this paper are based on the results presented by the authors at III TTOS Conference in May 2015.*

***KEYWORDS**: TECHNICAL SYSTEMS WITH UNCERTAIN PARAMETERS, INVERSE PROBLEM OF STABILITY, NUMERICAL ALGORITHM*

1. Introduction

One of the main concerns of analysis of properties of dynamic system (technical, economic, biological, etc.), or design of such systems is an appraisal of the conditions for the preservation of the properties of the system when small changes to the system parameters occur. In particular, this information is important in assessing the degree of maintainability of the system when changes to various structural components of the system occur. Due to the complexity and sometimes impossibility to provide the necessary and sufficient allowable ranges of the respective parameters, presentation of at least a sufficient assessment can be of great interest. On the other hand, experience shows that availability of universal theoretical results, as a rule, leads to great difficulties in applying these results to solve specific problems. We believe that such considerations should be taken into account when preparing theoretical structures to address specific practical problems. However, this line of thought does not exclude the use of formal logic as the basis of the arguments, which relate to the application of mathematical methods and designs.

Suppose that the preservation of a system's property (properties)), which we are interested to preserve, is determined by the requirement of fulfilling the following *m* conditions with respect to *n* parameters $\{\varepsilon_j\},\ j=1,2,...,n$, associated with this system:

$$f_i(\varepsilon_1,\varepsilon_2,...,\varepsilon_n)<0,\ i=1,2,...,m\ . \qquad (1)$$

Assuming, that each function $f_i(\varepsilon_1,\varepsilon_2,...,\varepsilon_n)$ is continuous (as will be seen below, it can even be considered as continuously differentiable) in the relevant *field* $\Omega_i\subset R^n$, and

$$\vec{0}\in\bigcap_{i=1}^{m}\Omega_i,\ \vec{0}\equiv(0,0,...,0)\in R^n\ . \qquad (2)$$

Also suppose that for all f_i in the point $\vec{0}$ the inequality $f_i(\vec{0})\le-\delta$ exist for $\delta>0$.

Thus there is an *n*-dimensional sphere D_ρ of some non-zero radius ρ centered at $\vec{0}$ for all points of which the inequality listed below is fulfilled:

$$f_i(\varepsilon_1,\varepsilon_2,...,\varepsilon_n)<-\delta/2,\ i=1,2,...,m\ . \qquad (3)$$

Though an n-dimentional cube *Q* with the center coinciding with the center of the *n*-dimentional sphere can be placed into the sphere. Thus the edges of the cube are parallel to the coordinate axes. The next statement is based on this fact.

Lemma 1. Let $(-\hat{\varepsilon}_i,\hat{\varepsilon}_i),\ \hat{\varepsilon}_i>0$ is the interval, for which all points ε are the solutions of the inequality $f_i(\varepsilon,\varepsilon,...,\varepsilon)<-\delta/2$. Then *n*- dimentional cube

$$Q=\{(\varepsilon_1,\varepsilon_2,...,\varepsilon_n):\varepsilon_j\in(-\hat{\varepsilon},\hat{\varepsilon})=\bigcap_{i=1}^{m}(-\hat{\varepsilon}_i,\hat{\varepsilon}_i);\ j=1,2,...,n\}$$ is

the one of the solutions of the system of inequalities (3).

This lemma allows us to propose the following method (seethe Section 2 below) to solve the problem of sufficient conditions for the preservation of stability of a linear dynamic system stability properties when possible (previously unknown, and in particular, random) changes to its parameters occur. The need for such evaluation may be a critical issue, for example, in a situation of replacement of certain structural elements of the technical system during its repair: when the ranges of values for the components being substituted and the ones that are used as substitution are always not known precisely.

2. Preserving stability of a linear dynamic system with uncertain perturbations of its parameters

So, let's assume that a dynamic system is given by the following system of differential equations:

$$\frac{dZ(t)}{dt}=AZ(t),\qquad\qquad (4)$$
$$A=(a_{ij})_{i,j=1}^{n},\ Z(t)=(z_1(t),\ z_2(t),...,z_n(t))^T.$$

With *A* is a known constant matrix and all its eigenvalues $\lambda_j,\ j=1,2,...,n$ are different and their real part is negative. In this case, as it is know, the system (4) is stable. However, if the matrix *A* is replaced with matrix $\tilde{A}=A+E,\ E=(\varepsilon_{ij})_{i,j=1}^{n}$, where ε_{ij} are unknown «perturbations» of the elements of the original matrix, then the question of the stability of a "perturbed" system $\frac{dZ(t)}{dt}=\tilde{A}Z(t)$ becomes relevant, and can be resolved on the basis of Lemma 1. In this case you must first prove that the eigenvalues $\lambda_k(E) \equiv \lambda_k(\varepsilon_{ij}),\ k = 1,2,...,n$ of matrix $\tilde{A} = A + E$ are sufficiently smooth functions of the parameters $\varepsilon_{ij},\ i, j = 1,2,...,n$ in the vicinity of zero. For the case of a single disturbance parameter, this problem was solved in the second half of the XXth century, primarily in the works of T. Kato (see, for example, [1]). For the case of several disturbance parameters, the problem was much tougher, and substantive results were achieved mainly in the works of Ji-guang Sun [2, 3, 4]. It is these last results, that we will use hereinafter.

Lemma 2. [2, 3, 4]. Let $\vec{p} \in C^N$, $A(\vec{p}) \in C^{n\times n}$ be a real analytic function of $\vec{p}$ in some vicinity of $U(\vec{0})$ of the origin, and $A(\vec{0})$ is symmetrical. Suppose that λ_i is a simple eigenvalue of $A(\vec{0})$, and $\vec{x}_i$ is associated eigenvectors satisfying the relations $A(\vec{0})\vec{x}_i = \lambda_i \vec{x}_i$, $\vec{x}_i^T A(\vec{0}) = \lambda_i \vec{x}_i^T$. Then:

1) There exits a simple eigenvalue $\lambda_i(\vec{p})$ of $A(\vec{p})$, which is an analytic function of $\vec{p}$ in some vicinity $U(\vec{0})$ of the origin, and $\lambda_i(\vec{0}) = \lambda_i$;

2) The eigenvector $\vec{x}_i(\vec{p})$ of matrix $A(\vec{p})$ corresponding to the eigenvalue may be defined to be analytic function of $\vec{p}$ inside $U(\vec{0})$, and $\vec{x}_i(\vec{0}) = \vec{x}_i$.

The proof of this lemma uses the fact that there is a matrix $X_{2i} \in C^{n\times(n-1)}$ such that the matrix $X = (\vec{x}_i, X_{2i})$ is not singular and satisfies the following equations

$$X^T X = I_n\ ,\quad X^T A(\vec{0}) X = \begin{pmatrix} \lambda_i & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & A_{2i} \end{pmatrix} \tag{5}$$

Where I_n is the identity matrix of size $n\times n$, and A_{2i} - is the identity matrix of size $(n-1)\times(n-1)$.

We recall that in this paper we can consider $A(\vec{0})$ a symmetric matrix. Below is a variant of constructing of the corresponding matrix X.

It is easy to show that for X we can take the matrix $X = (\vec{x}_i, X_{2i})$, where $\vec{x}_i$ a *unit* eigenvector of symmetric matrix $A \in C^{n\times n}$, corresponding to the eigenvalue λ_i, and the columns of the matrix $X_{2i} \in C^{n\times(n-1)}$ are vectors which are orthogonal to vector $\vec{x}_i$ and *aren't eigenvectors of matrix A*.

Indeed, let's take a closer look at the expression $X^T A$:

$$X^T A = \begin{pmatrix} \vec{x}_i & \vec{y}_1 & ... & \vec{y}_{n-1} \end{pmatrix}^T \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & ... & ... & a_{nn} \end{pmatrix}, \tag{6}$$

where $a_{ij} \in R$, $\vec{x}_i$ is the eigenvector corresponding to eigenvalue λ_i, and vectors $\vec{y}_j$, $j = 1,...,(n-1)$ are the vectors, which are orthogonal to vector $\vec{x}_i$. Since the matrix A is symmetric, and $\vec{x}_i$ is its eigenvectors, then the condition $\vec{x}_i^T A = \lambda_i \vec{x}_i^T$ is true. Therefore, it is easy to see that the expression (6) will be as follows:

$$X^T A = \begin{pmatrix} \lambda_i \vec{x}_i^T \\ A^* \end{pmatrix}, \tag{7}$$

where $\lambda_i \vec{x}_i^T$ is a row vector, and $A^* \in C^{(n-1)\times n}$. Thus

$$X^T A X = \begin{pmatrix} \lambda_i \vec{x}_i^T \\ A^* \end{pmatrix} \begin{pmatrix} \vec{x}_i & \vec{y}_1 & ... & \vec{y}_{n-1} \end{pmatrix} = \begin{pmatrix} \lambda_i \cdot (\vec{x}_i, \vec{x}_i), \lambda_i \cdot (\vec{x}_i, \vec{y}_1), ..., \lambda_i \cdot (\vec{x}_i, \vec{y}_{n-1}) \\ A_2^* \end{pmatrix}, \tag{8}$$

Here $(\vec{x}, \vec{y})$ is a scalar product. Due to the fact that $\|\vec{x}_i\| = 1$, $(\vec{x}_i, \vec{y}_j) = 0$ by virtue of the respective orthogonal vectors and the matrix $X^T A X$ is symmetrical by the symmetry of the matrix A, the expression (8) takes the following form:

$$X^T A X = \begin{pmatrix} \lambda_i & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & A_{2i} \end{pmatrix}, \tag{9}$$

q.e.d..

We have the following fundamental result:

Theorem (Ji-g. Sun, [3, 4, 5]). Let's suppose that λ_i is a simple eigenvalue of $A(\vec{0})$, and x_i is associated to its eigenvectors that satisfy $\|x_i\| = 1$. Let $\vec{\varepsilon} \equiv (\varepsilon_{11}, ..., \varepsilon_{1n}, ..., \varepsilon_{n1}, ..., \varepsilon_{nn})^T$, and $A(\vec{\varepsilon}) = A(\vec{0}) + E$, where the matrix "perturbation" is $E = (\varepsilon_{ij})_{i,j=1}^n$. Then matrix $A(\vec{\varepsilon})$ corresponding to the conditions of Lemma 2 and the following statement is true:

$$\lambda_i(\vec{\varepsilon}) = \lambda_i + \vec{x}_i^T E \vec{x}_i + \vec{x}_i^T E X_{2i} (\lambda_i I - A_{2i})^{-1} X_{2i}^T E \vec{x}_i + O(\|E\|^3), \tag{10}$$

where matrix X_{2i} is defined early.

This result allows us to apply in this situation the argument listed in the Section 1 of this paper, i.e. take the first three terms on the right side of the formula (10) of this work, as a function $f_i(\vec{\varepsilon})$.

We demonstrate the application of the methodology described above in the example shown in the Section 3 of this paper.

3. A numerical example

Let matrix $A(\vec{0})$ take the form

$$A(\vec{0}) = \begin{pmatrix} -1{,}27 & 0{,}25 & 1 & 0 \\ 0{,}25 & -3{,}16 & 0 & 1{,}1 \\ 1 & 0 & -2{,}6 & 1 \\ 0 & 1{,}1 & 1 & -6{,}2 \end{pmatrix}, \tag{11}$$

and the "perturbation" matrix is

$$E = \begin{pmatrix} \varepsilon_{11} & 0 & 0 & \varepsilon_{14} \\ \varepsilon_{21} & \varepsilon_{22} & 0 & 0 \\ \varepsilon_{31} & 0 & \varepsilon_{33} & 0 \\ 0 & \varepsilon_{42} & 0 & 0 \end{pmatrix}$$

The vector of eigenvalues of matrix (11) equates to $\vec{\lambda} = (\text{-6,7049} \quad \text{-3,0383} \quad \text{-2,7402} \quad \text{-0,6466})^T$. All the eigenvalues are simple. The real part of the eigenvalues in the left part is a coordinate axis and, consequently, the unperturbed system (4) is stable..

Applying the method proposed above for estimating the value of perturbations of matrix *A*, during which the stability property of the perturbed matrix $A(\vec{\varepsilon}) = A + E$ is preserved. Here the appropriate algorithm is demonstrated on the example of the first eigenvalue $\lambda_1 = \text{-6,7049}$. The corresponding its eigenvector is $\vec{x}_1 = (0{,}0574 \quad \text{-0,2909} \quad \text{-0,2392} \quad 0{,}9246)^T$.

Matrix $X_1 = (\vec{x}_1, X_{21})$:

$$X_1 = \begin{pmatrix} 0{,}0574 & 1 & 0 & 0 \\ \text{-0,2909} & 0 & 1 & 0 \\ \text{-0,2392} & 0 & 0 & 1 \\ 0{,}9246 & 0 & 0 & 0 \end{pmatrix} \tag{12}$$

Applying to the Gram-Schmidt orthogonalization to the matrix (12), we obtain for X_1:

$$X_1 = \begin{pmatrix} 0.057 & -0.998 & 0 & 0 \\ -0.291 & -0.017 & 0.957 & 0 \\ -0.239 & -0.014 & -0.073 & 0.968 \\ 0.925 & 0.053 & 0.282 & 0.25 \end{pmatrix}$$

Now we can find the matrix $X_1^T A(0) X_1$:

$$X^T A(0) X = \begin{pmatrix} -6.7049 & 0 & 0 & 0 \\ 0 & -1.252 & -0.1664 & -0.9697 \\ 0 & -0.1664 & -2.8385 & 0.271 \\ 0 & -0.9697 & 0.271 & -2.3346 \end{pmatrix},$$

where the block of size 3x3 is the matrix A_{21}.

It is now possible to use the formula (10) for drawing up the inequalities of the type (3).

The following expression of the form (10) can be compiled for $\lambda_1(\vec{\varepsilon})$:

$$\lambda_1(\vec{\varepsilon}) = 0.00329\varepsilon_{11} + 0.0531\varepsilon_{14} - 0.0167\varepsilon_{21} + 0.0846\varepsilon_{22} -$$
$$-0.0137\varepsilon_{31} - 0.269\varepsilon_{42} + 0.0572\varepsilon_{33} - 0.01\varepsilon_{11}\varepsilon_{14} +$$
$$+0.0137\varepsilon_{11}\varepsilon_{21} - 0.00006\varepsilon_{11}\varepsilon_{22} + 0.0509\varepsilon_{14}\varepsilon_{21} +$$
$$+0.0419\varepsilon_{14}\varepsilon_{31} - 0.0076\varepsilon_{14}\varepsilon_{33} - 0.0006\varepsilon_{21}\varepsilon_{31} -$$
$$-0.0002\varepsilon_{11}\varepsilon_{42} - 0.0004\varepsilon_{22}\varepsilon_{31} + 0.0022\varepsilon_{21}\varepsilon_{33} +$$
$$+0.0045\varepsilon_{22}\varepsilon_{33} + 0.0021\varepsilon_{14}\varepsilon_{42} - 0.0107\varepsilon_{21}\varepsilon_{42} +$$
$$+0.0583\varepsilon_{22}\varepsilon_{42} + 0.0052\varepsilon_{31}\varepsilon_{33} + 0.0032\varepsilon_{31}\varepsilon_{42} -$$
$$-0.0107\varepsilon_{33}\varepsilon_{42} - 0.0006\varepsilon_{11}^2 + 0.0011\varepsilon_{14}^2 - 0.00003\varepsilon_{21}^2 -$$
$$-0.0201\varepsilon_{22}^2 - 0.00051\varepsilon_{31}^2 - 0.0129\varepsilon_{33}^2 + 0.0178\varepsilon_{42}^2$$

The next step based on Lemma 1 is replacing all ε_{ij} to ε :

$$\lambda_1(\varepsilon) = 0{,}1226\varepsilon^2 - 0{,}1012\varepsilon - 6{,}7049 .$$

Doing the same for the rest of the eigenvalues, we obtain the following equations:

$$\lambda_2(\varepsilon) = -0{,}2333\varepsilon^2 + 0{,}9831\varepsilon - 3{,}0383,$$
$$\lambda_3(\varepsilon) = -0{,}1254\varepsilon^2 + 0{,}4785\varepsilon - 2{,}7401,$$
$$\lambda_4(\varepsilon) = -0{,}0146\varepsilon^2 + 1{,}6395\varepsilon - 2{,}7401.$$

Solving the system of inequalities $\{\lambda_i(\varepsilon) < 0\}, i = 1,2,...,4$, we obtain a sufficient condition for preserving the stability of the perturbed system in the form of:

$$\varepsilon_{jk} \in (-6.995,\ 0.395),\ j,k = 1,2,3,4$$

However, given that equation (10) suggests a relative smallness of the disturbance matrices *E*: $\|E\|^3 \ll \|E\|^2$, the interval of possible values of the parameters ε_{jk} has to be specified, for example, such as $\varepsilon_{jk} \in (-0.05,\ 0.05),\ j,k = 1,2,3,4$.

4. Conclusion

The paper presents and substantiates the method for estimating acceptable ranges of small perturbations of several parameters of a dynamic system, which ensure the preservation of stability of the system.

5. References

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