EULER BERNOULLI THEORY FOR A 3-DIMENSIONAL, VARIABLE-CURVATURE BEAM

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Abstract: The linear theory including the effects of bending-torsion coupling and rotatory inertia is used to derive the equations of motion for a space beam with variable curvature. The governing differential equations of motion are derived based on Euler-Bernoulli beam theory via Hamilton’s principle. The full, coupled system of governing partial differential equations has a total order of 12.

Keywords: EULER BERNOULLI THEORY, VARIABLE-CURVATURE BEAM

1. Introduction

While many of the curved elements used in engineering structures are very common shapes like circular arcs and helixes, arbitrarily shaped elements also find usage which may be expected to expand with the ever increasing complexity of mechanisms, aerospace and civil structures. The general equations for the small vibrations of an arbitrarily-shaped space beam were derived long ago; the definitive reference for this (and many other problems) is Love (1944) [1] according to whom the general formulation for arbitrarily-curved space beams is due to Clebsch (1862) [2].

2. Governing Equations

Following Love [1], we denote the axis along the arc length of the spatial curve formed by the centroids of the cross-sections (which will be called the wire axis) as z. X and y axes are chosen to be the principal axes of the cross-section (Fig. 1).

![Frenet vectors and coordinate system attached to beam axis](image)

Fig. 1 Frenet vectors and coordinate system attached to beam axis

The xyz system is therefore attached to the wire axis with z denoting the tangent direction while x and y representing the orientation of the cross-section with respect to z. The principal normal (N) and the binormal (B) at any point on the wire axis are within the cross-section, however, they do not coincide with x and y axes, in general. We denote the angle between N and y as γ. The change in γ as one moves along the main helix axis, dγ / ds is termed the torsional twist, where s is the arclength along z. The unit tangent along z is denoted T. These are related by the Frenet equations:

\[ \frac{dT}{ds} = \Delta \times T \]  
\[ \frac{dN}{ds} = \Delta \times N \]  
\[ \frac{dB}{ds} = \Delta \times B \]

where \( \Delta = \tau T + \kappa B \) is the Frenet vector. It is the angular velocity of the TNB system as its origin moves with unit velocity along the wire axis. The xyz system differs from the TNB system by a rotation around T (z axis) through the angle γ between N and y; therefore

\[ i = \sin \gamma N - \cos \gamma B \]  
\[ j = \cos \gamma N + \sin \gamma B \]  
\[ k = T \]

where \( i, j, k \) are the unit vectors along xyz. The changes in these can be expressed as

\[ \frac{di}{ds} = \omega \times i = \lambda j - \kappa \gamma k \]  
\[ \frac{dj}{ds} = \omega \times j = -\lambda i - \kappa x k \]  
\[ \frac{dk}{ds} = \omega \times k = \kappa \gamma i - \kappa x j \]

where

\[ \omega = \Delta + \frac{dy}{ds} k \]
is the angular velocity of the xyz system as its origin moves with unit velocity along the wire axis; it differs from the Frenet vector by torsional twist. In the xyz system

$$\omega = \kappa_x i + \kappa_y j + \lambda k$$

(6)

where

$$\kappa_x = -\kappa \cos \gamma$$

(7a)

$$\kappa_y = \kappa \sin \gamma$$

(7b)

are the curvatures around x and y directions, and

$$\lambda = \frac{dy}{ds} + \tau$$

(8)

is the total twist with first term showing the torsional and the second the geometric twist. Also relations between geometric quantities are needed, i.e., between curvatures and twist before and after the deformation:

$$\omega = \kappa_0 x + \kappa_0 y$$

(13a)

$$\lambda = \frac{d\lambda}{ds}$$

(13b)

are given by Eqs. (13a) - (13b). The deformation vector is denoted as

$$U = Ui_0 + Vj_0 + WK_0$$

(9)

The point, initially located at $$\mathbf{R}$$, moves to

$$\mathbf{r} = \mathbf{R} + \mathbf{U}$$

(10)

Differentiating this expression with respect to the arclength

$$k = k_0 + \frac{dU}{ds}$$

(11)

where $$k = dR / ds$$, $$k_0 = dR / ds$$. Since the bar is assumed to be unextended, it does not matter what arclength is meant by s. The relation between the base vectors of undeformed and deformed coordinate systems is written as

$$i = L_1 i_0 + M_1 j_0 + N_1 k_0$$

(12a)

$$j = L_2 i_0 + M_2 j_0 + N_2 k_0$$

(12b)

$$k = L_3 i_0 + M_3 j_0 + N_3 k_0$$

(12c)

We find, from Eq. (12c),

$$L_3 = \frac{dU}{ds} - \tau_0 V + \kappa_y W$$

(13a)

$$M_3 = \frac{dV}{ds} - \kappa_x W + \tau_0 U$$

(13b)

$$N_3 = 1 + \frac{dW}{ds} - \kappa_y U + \kappa_x V$$

(13c)

$$L_3$$ and $$M_3$$ are $$O(\|U\|)$$ (assuming deformation and deformation gradient are of the same small order) while $$N_3$$ is $$O(1)$$. Since $$\|k\|^2 = L_3^2 + M_3^2 + N_3^2 = 1$$, substituting from Eq. (A.16) and ignoring $$O(\|U\|)$$ terms,