ONE ALGORITHM FOR SOLVING THE INVERSE PROBLEM OF STABILITY OF A DYNAMICAL SYSTEM WITH UCERTAIN PARAMETERS AND ITS APPLICATION ON THE EXAMPLE OF PREDICTING THE COURSE OF CERTAIN PROCESSES IN CHEMICAL REACTORS

ОДИН АЛГОРИТМ РЕШЕНИЯ ОБРАТНОЙ ЗАДАЧИ УСТОЙЧИВОСТИ ДИНАМИЧЕСКОЙ СИСТЕМЫ С НЕОПРЕДЕЛЕННЫМИ ПАРАМЕТРАМИ И ЕГО ПРИМЕНЕНИЕ НА ПРИМЕРЕ ПРОГНОЗИРОВАНИЯ ПРОТЕКАНИЯ НЕКОТОРЫХ ПРОЦЕССОВ В ХИМИЧЕСКИХ РЕАКТОРАХ

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Abstract: *The task of estimating the ranges of permissible changes in the design parameters of a chemical reactor is solved in which the stability property of the monomerization process is retained in it. The solution is based on Ji-guang Sun's results about the perturbation of the spectrum of the matrix, the elements of which depend on several indeterminate parameters.*
***KEYWORDS**: TECHNICAL SYSTEMS WITH UNCERTAIN PARAMETERS, INVERSE PROBLEM OF STABILITY, NUMERICAL ALGORITHM*

1. Introduction

One of the main concerns of dynamic systems property analysis is an appraisal of the conditions for the preservation of the system properties when small changes to the system parameters occur. For example, this information is important in assessing the degree of maintainability of the system when changes to various structural system components occur. Due to the complexity and sometimes impossibility to provide the necessary and sufficient allowable ranges of the respective parameters, presentation of at least a sufficient assessment can be great interest. On the other hand, experience shows that availability of universal theoretical results leads to great difficulties in applying these results to solve specific issues. We believe that such considerations should be taken into account during preparing theoretical structures for solving specific practical problems

Let to formalize the statement of the problem. Consider a dynamic system:

$$\frac{dZ(t)}{dt} = A(\mathrm{E})Z(t),$$
$$A = (a_{ij}(\mathrm{E}))_{i,j=1}^{n},\ Z(t) = (z_1(t), z_2(t), ..., z_n(t))^T,\ \mathrm{E} = (\varepsilon_1, ..., \varepsilon_m). \qquad (1)$$

Suppose that the preservation of the system's (1) property (properties) , which we are interested to preserve, is determined by the requirement of fulfilling the following *k* conditions with respect to *m* parameters $\{\varepsilon_j\},\ j = 1, 2, ..., m$, associated with this system:

$$f_i(\varepsilon_1, \varepsilon_2, ..., \varepsilon_m) < 0,\ i = 1, 2, ..., k \qquad (2)$$

Lemma 1. [1] Let $(-\hat{\varepsilon}_i, \hat{\varepsilon}_i),\ \hat{\varepsilon}_i > 0$ is the interval, for which all points ε are the solutions of the inequality $f_i(\varepsilon, \varepsilon, ..., \varepsilon) < -\delta/2$. Then *m* - dimensional cube

$$Q = \{(\varepsilon_1, \varepsilon_2, ..., \varepsilon_m) : \varepsilon_j \in (-\hat{\varepsilon}, \hat{\varepsilon}) = \bigcap_{i=1}^{k} (-\hat{\varepsilon}_i, \hat{\varepsilon}_i);\ j = 1, 2, ..., m\}$$ is

the one of the solutions of the system of inequalities (2).

2. Stability problem of a linear dynamical system with small perturbations of its parameters

Let's assume that a dynamic system is given by the following system of differential equations:

$$\frac{dZ(t)}{dt} = AZ(t),$$
$$A = (a_{ij})_{i,j=1}^{n},\ Z(t) = (z_1(t), z_2(t), ..., z_n(t))^T. \qquad (3)$$

Where A is a known constant matrix and all its eigenvalues $\lambda_j,\ j = 1, 2, ..., n$ are different and their real parts are negative. In this case the system (3) is stable. However, if the matrix *A* is replaced with matrix $\tilde{A} = A + E,\ E = (\varepsilon_{ij})_{i,j=1}^{n}$, where ε_{ij} are unknown «perturbations» of the original matrix elements, then the question of the stability of a "perturbed" system $\frac{dZ(t)}{dt} = \tilde{A}Z(t)$ becomes relevant and can be resolved by using Lemma 1. So, first, the fact that the eigenvalues $\lambda_k(E) \equiv \lambda_k(\varepsilon_{ij}),\ k = 1, 2, ..., n$ of matrix $\tilde{A} = A + E$ are sufficiently smooth functions of the parameters $\varepsilon_{ij},\ i, j = 1, 2, ..., n$ in the neighborhood of zero must be proved. For the case of a single disturbance parameter, this problem was solved in the second half of the XXth century, primarily in the works of T. Kato (see, for example, [2]). For the case of several disturbance parameters, the problem was much tougher, and substantive results were achieved mainly in the works of Ji-guang Sun [3-7]. These last results we shall use below.

In [3, Theorem 2.2] Ji-guang Sun uses the following statement (without proof):

Theorem 1. Let λ be a simple eigen value of matrix $A \in \mathbb{C}^{n\times n}$, $\vec{x}, \vec{y}$ are corresponding left and right eigenvectors, $\vec{x}^T \vec{y} = 1$. Then for each λ there are $\tilde{X}$, $\tilde{Y} \in \mathbb{C}^{n\times(n-1)}$: $X = (\vec{x}, \tilde{X})$ and $Y = (\vec{y}, \tilde{Y})$, such that:

$$X^T Y = I_n, \qquad (4)$$

$$X^T AY = \begin{pmatrix} \lambda & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & \tilde{A} \end{pmatrix}. \qquad (5)$$

We will show a constructive proof of this statement which allows to design a simple algorithm of numerical solution to an inverse problem of dynamic system stability with uncertain parameters.

Assume that $\tilde{X}$ is an orthonormal matrix and its columns are orthogonal to a vector $\vec{x}$, $\tilde{Y}$ is a matrix which consists of X^{-1} columns: $\tilde{Y} = \{h_{i,j}\}_{i=1,j=2}^{i=n,j=n}$, where $H \square \{h_{i,j}\}_{i=1,j=1}^{i=n,j=n} = X^{-1}$.

Since the following statement is true:

$$X^T Y = \left(\vec{x}, \tilde{X}\right)^T \left(\vec{y}, \tilde{Y}\right) = \begin{pmatrix} \vec{x}^T \vec{y} & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & I_{n-1} \end{pmatrix}$$

and columns of $\tilde{X}$ are orthogonal to $\vec{y}$, columns of $\tilde{Y}$ are orthogonal to $\vec{x}$, the (3) is also true.

Consider the following expression:

$$X^T A = (\vec{x}, \vec{z}_1, \vec{z}_2, \ldots, \vec{z}_{n-1})^T (\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n) = \\ = \begin{pmatrix} \vec{x}^T \vec{a}_1 & \vec{x}^T \vec{a}_2 & \ldots & \vec{x}^T \vec{a}_n \\ & A^* & & \end{pmatrix} = \begin{pmatrix} \vec{x}^T A(\vec{0}) \\ A^* \end{pmatrix} = \begin{pmatrix} \lambda \vec{x}^T \\ A^* \end{pmatrix}, \qquad (6)$$

где $\vec{a}_j$, $j = 1, 2, \ldots, n$ - columns of matrix A, $A^* \in \square^{(n-1)\times n}$.

From (6) we obtain:

$$X^T AY = \begin{pmatrix} \lambda \vec{x}^T \\ A^* \end{pmatrix} (\vec{y}, \vec{h}_1, \vec{h}_2, \ldots, \vec{h}_{n-1}) = \\ = \begin{pmatrix} \lambda \vec{x}^T \vec{y} & \lambda \vec{x}^T \vec{h}_1 & \ldots & \lambda \vec{x}^T \vec{h}_{n-1} \\ & A^{**} & & \end{pmatrix} = \begin{pmatrix} \lambda & 0_{1\times n-1} \\ & A^{**} \end{pmatrix}, \qquad (7)$$

where $\vec{x}^T \vec{y} = 1$ and $\vec{x}^T \vec{h}_i = 0$.

Also, let's use the following expression:

$$AY = \begin{pmatrix} \vec{a}^1 \\ \vec{a}^2 \\ \vdots \\ \vec{a}^n \end{pmatrix} \cdot \left(\vec{y}, \vec{h}_1, \vec{h}_2, \ldots, \vec{h}_{n-1}\right) = \left(\lambda \vec{y}, \quad A'\right), \qquad (8)$$

where $\vec{a}^j$ are rows of the matrix A. $A' \in \square^{n\times(n-1)}$

From (8) we obtain:

$$X^T AY = (\vec{x}, \vec{z}_1, \vec{z}_2, \ldots, \vec{z}_{n-1}) \cdot \left(\lambda \vec{y}, \quad A'\right) = \begin{pmatrix} \lambda & \\ 0_{(n-1)\times 1} & A'' \end{pmatrix}, \qquad (9)$$

because $\vec{x}^T \vec{y} = 1$ and $\vec{z}^T \vec{y} = 0$.

From associative property of matrix multiplication and expressions (8) and (9), we can conclude that statement (5) is true.□

According to [4], if conditions of Theorem 1 are satisfied the following decomposition takes place:

$$\lambda(\vec{\varepsilon}) = \lambda + \vec{y}^T E \vec{x} + \vec{y}^T E X_2 (\lambda I - A_2)^{-1} Y_2^T E \vec{x} + O(\|E\|^3). \qquad (10)$$

3. *Algorithm for the numerical solution of the inverse stability problem for a dynamical system with uncertain parameters*

Using the proof of Theorem 1 of §2, we formulate an algorithm for the numerical solution of the inverse stability problem for a dynamical system with uncertain parameters.

Step 1. Compute the *single* eigenvalue λ of the matrix A and the corresponding left and right eigenvectors -- $\vec{x}, \vec{y}$ **.**

Step 3. Form the matrix $H = (\vec{y}_i, N)$, where

$$N = \begin{pmatrix} 1 & 0 & \ldots & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in C^{n\times(n-1)}.$$

Step 3. Orthogonalize the Gram-Schmidt matrix H. The result is an orthonormal matrix H_2.

Step 4. Form the matrixes X and Y by this way: $X = (\vec{x}_i, H_3)$, $Y = (\vec{y}_i, H_4^{-1})$, где $H_3 : \{h_3^{ij}\}_{i=1,j=1}^{i=n,j=n-1} = \{h_2^{ij}\}_{i=1,j=2}^{i=n,j=n}$, $H_4 : \{h_4^{ij}\}_{i=1,j=1}^{i=n,j=n-1} = \{x^{ij}\}_{i=1,j=2}^{i=n,j=n}$, i.e. the matrix H_3 is the matrix H_2 without the first column. And H_4 is the matrix X without the first column.

4. *Analysis of the stability of the monomerization process in a chemical reactor*

Consider the algorithm presented above for the example of the stability problem of the monomerization process in a chemical reactor [7].

$$\begin{cases} V\dfrac{dC}{dt} = F\left(C_0 - C\right) - Vk(T)C, \\ \rho c_\rho V\dfrac{dT}{dt} = F\rho c_\rho\left(T_0 - T\right) + Vk(T)C\Delta H - K_t B(T - T_c), \\ \rho_c c_{\rho c} V_c \dfrac{dT_c}{dt} = F_c \rho_c c_{\rho c}(T_{c0} - T_c) + K_t B(T - T_c) \end{cases} \qquad (11)$$

где $k(T) = Z \cdot \exp\left(\dfrac{-E}{RT}\right)$.

The parameters of this system are known [7, 8]. Note, from the mathematical point of view, the stability of the monomerization process is equivalent to the stability of the solutions of the system (11).

If to represent the system (11) in the form (3), then “unperturbed” matrix A for the considered process becomes:

$$A = \begin{pmatrix} \text{-476,144} & 0{,}179 & 0 \\ 123{,}445\cdot 10^3 & \text{-51,029} & 1{,}327\cdot 10^{-5} \\ 0 & 0{,}006 & \text{-2,376} \end{pmatrix} \qquad (12)$$

The “perturbation” matrix:

$$E = \begin{pmatrix} \varepsilon_1 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \\ 0 & \varepsilon_4 & 0 \end{pmatrix} \qquad (13)$$

Using the algorithm presented in the previous paragraph, let to calculate the eigenvalues of the matrix:

$$\vec{\lambda} = \left(\text{-523,072}; \quad \text{-4,101}; \quad \text{-2,376}\right) \qquad (14)$$

The real parts of the eigenvalues are less than zero, so the system (11) is stable.

To illustrate the algorithm use $\lambda_2 = -4{,}101$. The right and left eigenvectors are equal to:

$$\vec{x}_2 = \left(-0{,}999 \quad -0{,}003 \quad 2{,}94\cdot 10^{-8}\right)^T,$$

$$\vec{y}_2 = \left(-3{,}8\cdot 10^{-4} \quad -0{,}999 \quad 0{,}003\right)^T$$

The expression (4) takes the form:

$$X^T AY = \begin{pmatrix} -4{,}101 & 0 & 0 \\ 0 & \text{-12,442}\cdot 10^3 & 0{,}0859 \\ 0 & \text{-0,403} & \text{-565,254} \end{pmatrix},$$

Then to calculate (9) for $\lambda_2 = -4{,}101$ by using Lemma 1:

$$\lambda_2(\varepsilon) = 6{,}819 \cdot 10^{-6} \varepsilon^2 + 3{,}689 \cdot 10^{-4} \varepsilon - 4{,}1$$

Thus, after solving inequality $\lambda_2(\varepsilon) < -\frac{\delta^2}{2}, |\delta| << 1$, we get the acceptable ranges for variation of ε. The important moment is the fact that this ranges should be a sufficiently small number for the relation (10) to be satisfied.

5. Conclusion

The paper presents and substantiates the method for estimating acceptable perturbation ranges of several parameters of a dynamic system, which ensure the preservation stability in the system.

For convenience, the authors propose to nondimensionalize the system before applying the proposed algorithm. It allows to scale the considering system.

6. References

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