Analytical and numerical analysis of shafts' stress and strain states of the hydro-power unit

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Abstract: *The paper presents both the analytical and numerical procedure for the analysis of the stress and strain states of the hydropower unit of the hydropower plant. Numerical analysis allows a comprehensive view of the states of stress and strain of the observed shaft. The analytical procedure is very difficult to apply to structural elements with complex geometry such as the shaft of a hydropower unit. Analytical analysis of stress and strain states serves to test only some of the results obtained on the basis of numerical analysis. A comparative analysis of stress and deflection values was performed on some characteristic shaft cross sections. Analytical determination of deflection, due to the complexity of the shaft construction, was performed on the basis of Mohr's graph - analytical procedure. The numerical procedure is based on the application of the finite element method. The obtained results confirm the extremely good match of the results obtained by analytical and numerical procedures. Within the numerical analysis based on the given load and boundary conditions, ie. on the basis of the formed computational model of the shaft, the analysis of stress and strain, and displacement, was performed. The analysis of both normal and tangential stresses, as well as equivalent stresses according to the Von-Misses criterion, was performed. The places of maximum stresses are located. Therefore, it is concluded that the numerical analysis of the stress and strain state will give quality results if the computational model is well formed. This means that it is necessary to take into account all the discontinuities in the geometry of the shaft, which is characterized by a variable cross section. Certainly, there is no need to apply a very, very demanding analytical procedure for integrating the differential equation of an elastic line or a procedure based on Mohr's graph - analytical procedure when calculating the shaft.*

Keywords: *STRESS AND STRAIN STATE, STRESS STATE ANALYSIS, STRAIN STATE ANALYSI, HYDROPOWER PLANT SHAFT ANALYSIS*

1. Introduction

Structural parts of the hydropower plant in which the conversion of potential and kinetic energy of water into electricity is performed are hydraulic turbines, shafts and generators. Together they form a structural unit called a hydro-aggregate. Depending on the type, size and mode of operation of the hydropower plant, the selection of the mentioned structural elements is made. The degree of energy utilization that accumulates largely depends on their maintenance, both regular and periodic. Also the revitalization of hydro power units after a certain time in operation is of great importance.

Observed hydro power turbines used to transform the potential energy of water into kinetic energy are Pelton turbines. Two impellers and one generator are screwed to the hydro turbine shaft. Simplified, it is a beam with two overhangs, whose geometric representation with basic dimensions is given in Fig. 1. The shaft has a rectilinear geometric axis with a total weight of approximately 42.700 kg.

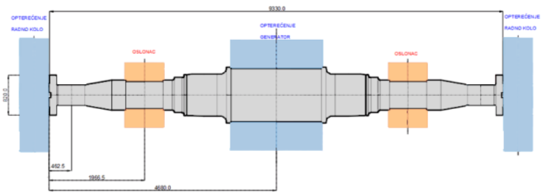

Fig. 1 *Simplified drawing of the analyzed shaft.(blue - loadings, red - supports)*

Basic characteristics of the hydropower plant's impeller are given in the Table 1.

Table 1: *Basic characteristics of the hydropower plant's impeller*

Turbine type	*Pelton*
Maximum drop [m]	*537.20*
Calculated drop [m]	*508*
Minimal drop [m]	*506*
Weight of the impeller [kg]	*cca 7500*
Turbine power in calc. drop [MW]	*58.5*
Diameter of the impeller [mm]	*2710*
Number of impeller blades	*18*
Rotational speed [rot/min]	*428.5*

Based on the analytical calculation, the shaft deflections and stress values were calculated in order to support the numerical procedure of shafts behavior diagnostics in order to check and compare the obtained results with the results obtained using the finite element method.

It is almost impossible to analyze a shaft of variable cross sections, such as a shaft of a hydro aggregate, based on the integration of the differential equation of the elastic line of the shaft. The differential equation of the elastic line is easier to solve using Mohr's graph-analytic method. Also, the stresses on individual shaft sections will be calculated.

2. Analytical calculation method

For more precise calculation, the shaft is divided into nine segments which are shown in the calculation drawing of the shaft given in Figure 2. Table 2 shows the values of forces acting on the shaft, and is shown in Figure 2. Due to symmetry with respect to the y axis only the first five forces are shown, as well as the forces acting due to the weights of the impellers and generators, and the reactive forces. Some of the basic technical characteristics together with the intensities of forces that replace the weight of the shaft by segments are given in Table 3.

Table 2. *Forces that act upon the shaft*

Loading forces of the shaft [kN]						
G_I	G_{II}	G_{III}	G_{IV}	G_V	G_T	G_G
8.55	34.42	21.62	76.35	136.87	73.57	1275.3

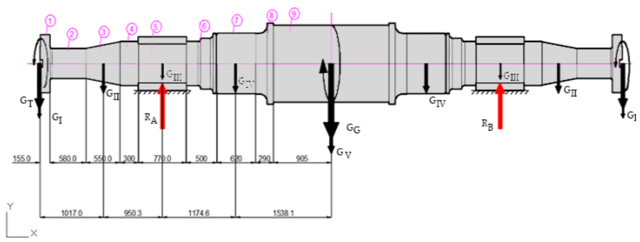

Fig. 2 *Calculation drawing of the shaft*

The weight of the impellers located at the ends of the shafts are 7.500 kg, while the weight of 130.000 kg was taken as the approximate weight of the generator rotor. Load values in the form of twisting moments are: 1.30 kNm at the turbine impeller locations (shaft ends) and 2.60 kNm at the generator location (shaft center).

Table 3. *Basic geometric characteristics of the shaft through shafts segments*

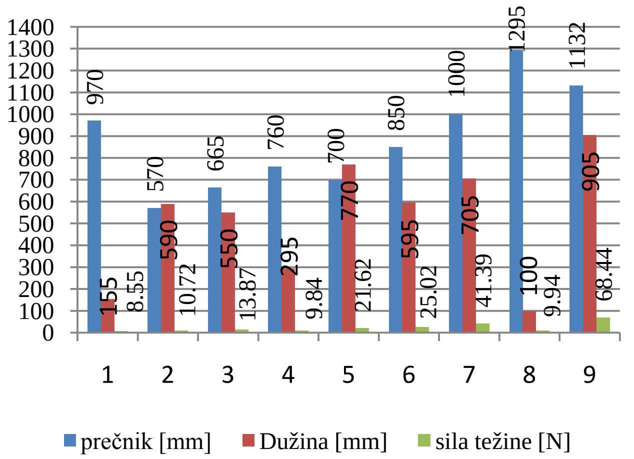

Table 4 [1] gives the necessary data for the calculation of the shaft of variable cross section by segments given in Figure 2.

Table 4. *Values of axial moment of inertia, reduction coefficients and product EIx*

Segm.	Axial moment of inertia Ix [m^4]	Reduction coefficient	Product EIx
1	0.04341	8.42	8681968
2	0.00515	1.00	1030840
3	0.00957	1.86	1913994
4	0.01634	3.17	3268696
5	0.01176	2.28	2351014
6	0.02559	4.96	5117219
7	0.04904	9.51	9807532
8	0.13796	26.77	27591834
9	0.08054	15.63	16107623

The sliding bearings A and B of the shaft allow rotation about the longitudinal z axis of the shaft, and all other movements and rotations of the shaft are prevented. Based on the above, the starting shaft can be separated into units: two consoles at the ends and a clamped beam in the middle of the shaft (Figure 3).

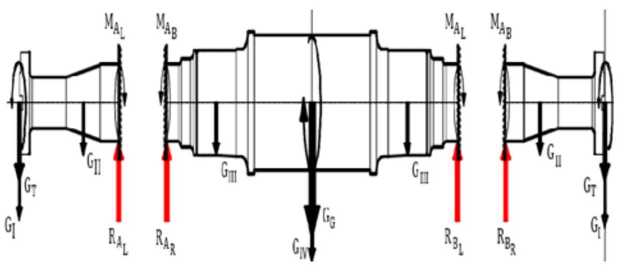

Fig. 3 *Shaft divided on wholes*

In order to verify the results based on the analytical procedure with the results obtained based on the numerical (FEM) procedure, the following will be calculated:

- deflection at the end of the shaft, ie. deflection at the end of the bracket in the variant of the split shaft (Fig. 4)

- normal bending stress at the clamping point of the bracket.

- **Deflection of the shaft at the end of the console calculus**

The differential equation of the elastic line of the beam for the case that the beam is of variable cross-section, due to the bending of the beam, as is the case with the shaft of the hydraulic unit is:

$$E\,I_x(z)y''=-M_{sx}(z), \qquad (1)$$

or

$$E\,I_{x,min}\,i(z)y''=-M_{sx}(z), \qquad (2)$$

where

$$I_x(z)=I_{x,min}\,i(z),$$

where $I_{x,min}$ is the smallest axial moment of inertia of the beam (shaft) of variable cross section. The coefficient i (z) ≥1 is called the bending moment reduction coefficient. Equation (2) by introducing the reduction coefficient takes the form:

$$E\,I_{x,min}y''=\frac{M_{sx}(z)}{i(z)}=M_{s,\,red}(z),$$

where $M_{s,red}\,(z)$ is the reduced bending moment. The problem of determining the deflection and inclination of the beam (shaft) of variable cross section is now reduced to determining the deflection and inclination of the beam of constant cross section of the minimum moment of inertia, but instead of the actual bending moment $M_{sx}(z)$ should use a reduced bending moment $M_{s,\,red}(z)$, which is obtained when the actual bending moment is reduced, ie. reduce and (z) times. In the case of beams, of the so-called stepwise cross-sections such as the shaft of the hydro aggregate for determining the deflection and slope is most often used Mohr's graph-analytical method [2]. It is known from the theory of bending of beams that:

$$\frac{d^2M_s(z)}{dz^2}=\frac{dF_T(z)}{dz}=-q(z), \qquad (3)$$

where q = q (z) is the continuous load of the beam. When q = q (z) one time integrals we get $F_T(z)$, and when $F_T(z)$ integrals we get $M_s(z)$. It is much more efficient to do these integrations graphically, rather than analytically, by drawing diagrams of the transverse force and the bending moment.

The differential equation of the elastic line of the beam, as already mentioned, can be written in the form:

$$\frac{d^2}{dz^2}(EI_xy)=-M_{s,\,red}(z), \qquad (4)$$

Observing equations (3) and (4) at the same time, it can be stated as follows. If $M_{s,red}\,(z)$ is understood as a continuous-fictitious load of the beam (shaft), then by drawing a diagram of the fictitious transverse force $\mathcal{F}=\mathcal{F}(z)$, we obtain the slope $y'(z)=\frac{\mathcal{F}(z)}{EI_x}$. When a diagram of the fictitious bending moment $\mathcal{M}=\mathcal{M}(z)$ is drawn, then the deflection of the real support $y(z)=\frac{\mathcal{M}(z)}{EI_x}$ can be determined. In this way, we essentially solved the differential equation graphically:

$$\frac{d^4}{dz^2}(EI_xy)=q(z),$$

ie. determined the deflection y (z) based on the real load q (z). From the expression for the deflection and inclination of the beam girder, it is clear that the dimension of the fictitious force is kNm^2, and the fictitious moment is kNm^3.

When applying the Mohr graph-analytical method, a fictitious beam must be observed instead of a real beam (shaft) (Figure 4). A fictitious beam must satisfy the boundary conditions.

At the point of clamping the console, ie. of the real beam girder, the deflection and inclination are equal to zero, and at the free end of the console the deflection and inclination are nonzero. From equations (3) and (4) it can be concluded that at the place where the deflection and the inclination are equal to zero, in the case of a fictitious beam the transverse force and the bending moment must be equal to zero. Similarly, in the place of a real beam where the deflection and the inclination are different from zero, the fictitious bending moment and the fictitious transverse force must be different from zero, i. in the case of a fictitious console, there must be a clamp (Fig. 4)

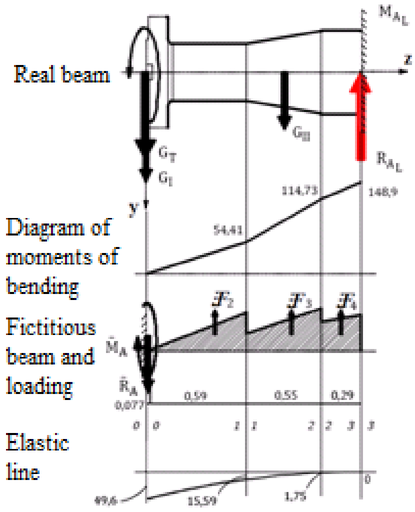

Fig. 4 *Diagrams of the bending moments, fictitious loads and elastic line of the beam*

The obtained values for both deflections (elastic line) and values of normal stresses will be used for comparison with the results obtained by numerical analysis.

3. Numerical calculation based on the FEM

- ***Calculation of the displacement***

The subject of the analysis of displacement using the finite element method is a shaft loaded with a combined stress. The shaft is loaded to bend around the axis transverse under the action of the shaft weight. Also, it is loaded on twisting under the action of twisting moments that act along the axis of the shaft. This analysis should answer the question of which areas are critical from the aspect of shaft displacement that is exposed to complex and combined stresses. By looking at the obtained results, it will be known in which sections the maximum deflections occur under the influence of the previously mentioned loads.

Loads and boundary conditions are given in Figure 5. Two sliding bearings shown in Figure 3 and marked with the letter A (purple) appear as supports. Only rotations about the z-axis (longitudinal) are allowed in the supports. Loads are marked in red and marked with letters B to J.

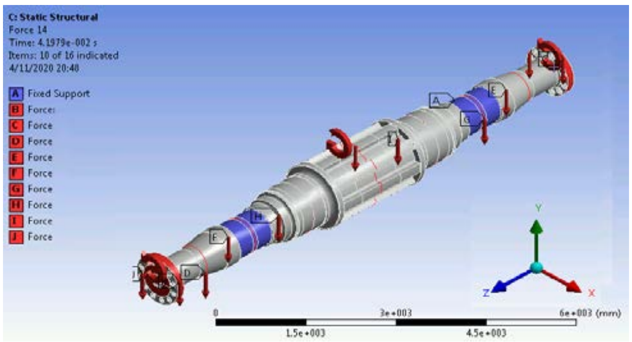

Fig 5. *Calculation model of the shaft with the supports and loads*

- Values of the displacements

The results of the numerical analysis are given as follows. The values of deflection in the y-axis direction can be seen in Figure 6. The characteristic values of deflection are given in the cross sections in which the forces act.

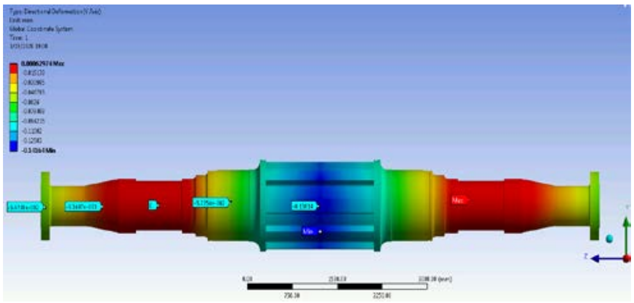

Fig. 6 *Values of the deflection in the direction of y axis*

The maximum value of the displacement in the y-axis direction under the action of forces that replace the masses of individual shaft segments, therefore, under the action of the shaft's own weight, is 0.142 mm. The curve of the deflection of the axis under the gravity of the shaft with the maximum and minimum values is given in Figure 7.

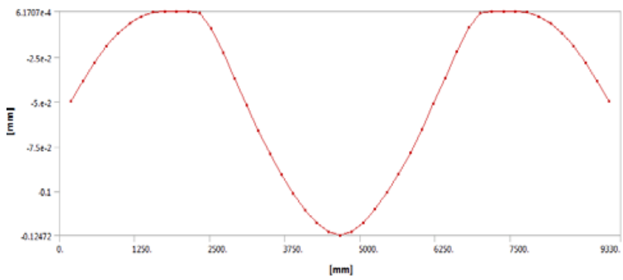

Fig 7. *Values of the deflection in the direction of y axis for the axis of the shaft which passes through the xy plane of symmetry of the shaft*

It can be seen in the diagram that the deflection values are equal to zero at the bearing locations, while the largest negative values, due to the orientation of the y axis, are at the points of action of external loads in the form of generators and impellers.

- ***Calculation of the stresses***

On the basis of numerical methods of research of the shaft stress state, it is possible to consider the shaft stress state as a part of hydro aggregates with greater accuracy and comprehensiveness in relation to analytical methods. With power transmissions, and especially with shafts, stress concentrations occur at the geometry discontinuity points. Cracks can often occur in these places, leading to loss of shaft function[3],[4],[5].

The stress state of the shaft, which is loaded by forces in the direction of the axis perpendicular to the shaft axis, and moments in the direction of the longitudinal axis of the shaft, is predominantly manifested by normal stress in the z direction of the σ_z axis, and tangential stresses τ_{zy} and τ_{zx}.

Figure 8 shows the state of normal stresses in the direction of the longitudinal z axis. Stresses in sections in which the forces of gravity of the segments by which the shaft is divided are separated. As can be seen in Figure 8, the value of the maximum stresses is 95 MPa much lower than the value of the allowable stress for the material from which the shaft is made (Steel with maximum allowed stress of 470 MPa).

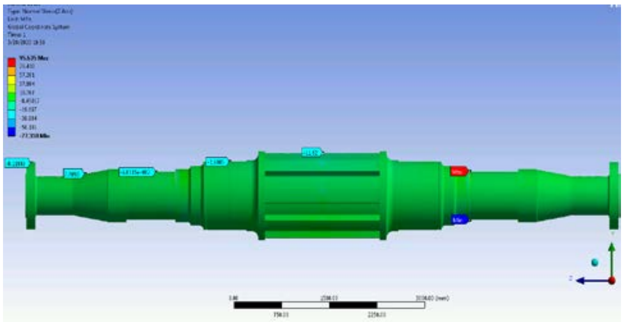

Fig 8. *Normal stresses in the direction of the z axis of the shaft*

Tangential stresses are predominantly caused by torsion moments acting on the shaft ends and in the middle of the shaft. Figure 9 and Figure 10 show the tangential stresses in the yz plane, i. τ_{zy} i τ_{zx}. It can also be seen, as in the case of normal stresses, that the values are much less than allowed, so it can be concluded that there is no risk of indications on the shaft due to static load.

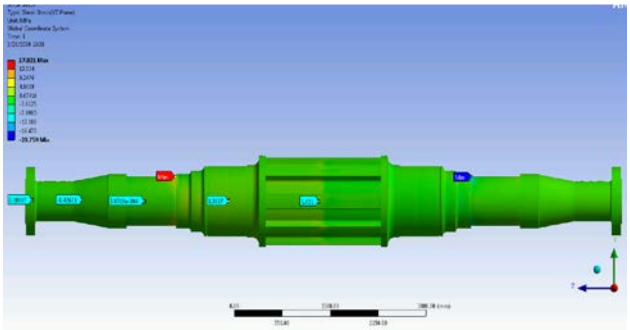

Fig 9. *Tangential stresses τ_{zy} in the plane of the z axis of the shaft*

Maximum (and minimum) stress values appear in the support zones, while in the cross sections where the forces (moments) act, stress values appear that are largely in the safe zone of elasticity.

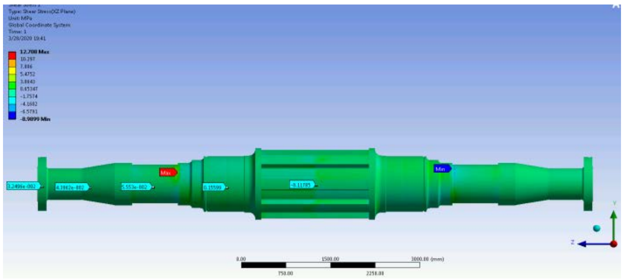

Fig 10. *Tangential stresses τ_{zx} in the plane of the z axis of the shaft*

Figure 11 shows the equivalent stresses calculated using Von-Mises - this criterion for calculating stresses at combined stresses. In this case, too, the numerical value of the maximum stresses is significantly below the limit value of the allowable stresses. Figure 12 shows the position of the maximum stresses, which corresponds to the position of the shaft bearing.

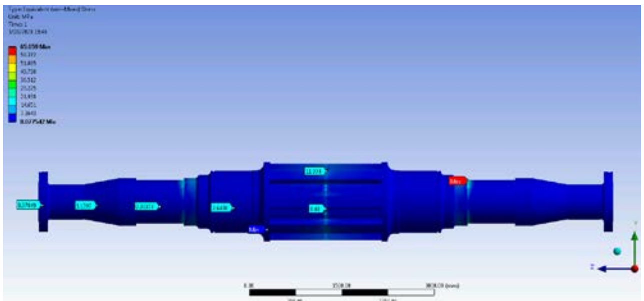

Fig 11. *Illustration of the equivalent stresses (Von - Mises criterion)*

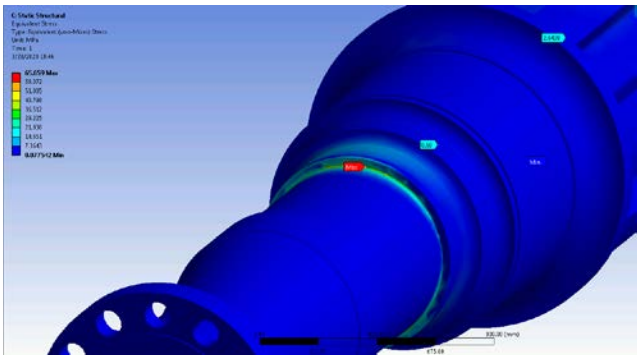

Fig 12. *Position of the maximum equivalent stresses (shafts bearing position)*

4. Conclusion

Based on the analytical and numerical (FEM) diagnostics of shaft behavior from the aspect of static load, the following can be stated:

- numeric values for deflection at the end of the shaft due to bending in the vertical plane were obtained,
- numeric values of normal stresses at the end of the shaft due to bending in the vertical plane were obtained (section 3-3, Figure 4),
- numeric values of normal stresses due to bending in the vertical plane were obtained (section 1-1, Figure 4).

The mentioned numeric values are compared and presented in Table 5.

Table 5. *Comparative values of analytical and numerical shaft calculation*

	Deflection [mm]	Normal stresses σ[MPa] (section 3-3)	Normal stresses σ [MPa] (section 1-1)
Analytical calculation	0.049	3.40	3.01
Numerical calculation	0.049	3.25	3.24

It can be concluded that the stresses, including the Von Mises stresses are evenly distributed along the entire shaft. The stress values, given in Table 5, refer to sections 1-1 and 3-3 from Figure 4. It is clear that they, as well as the stress values along the entire shaft, are well below the maximum stresses from the point of view of strength.

A similar conclusion can be drawn for the values of displacement in the direction of the y-axis (deflection). In the analytical calculation, the maximum value of deflection of 0.049 mm at the end of the shaft was obtained, which was confirmed by numerical analysis.

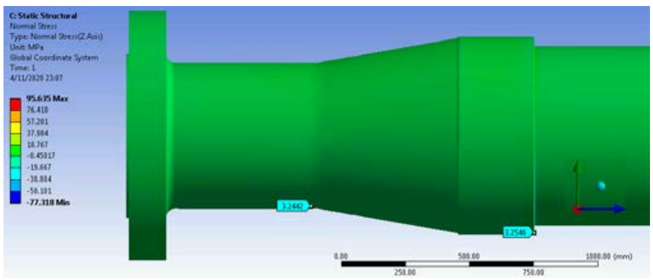

Fig 13. *Values of the normal stresses in the sections 1-1 and 3-3*

Dynamic analysis of the behaviour of the shaft is also very interesting problem which is not presented in this paper, but is a part of a wider research which is included in the reference [6].

5. References

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