Analytical Studies of the Positioning of the End-Effector of a Manipulator Robotic System

Oleh Chernysh
National University of Life and Environmental Sciences of Ukraine, Kyiv, Ukraine
E-mail: chernysh@nubip.edu.ua

Abstract. *The article presents an algorithm for solving the forward kinematics problem for a multi-link manipulator robotic system model based on the matrix method of attached coordinates.*
KEYWORDS: *MANIPULATOR, ROBOTIC SYSTEM, END-EFFECTOR POSITIONING.*

Introduction

The question of developing and implementing applied calculation methods within specific software environments, tailored to defined operational conditions, is highly relevant today. To construct the required algorithms for these complex calculations, engineers have the flexibility to utilize generally recognized, powerful software packages, such as MathCAD, MATLAB, or Maple, alongside other highly specialized software products.

In general, the meticulous selection of methods and means for programming and calculating the kinematic parameters of robotic systems is critically important across most modern technological processes [1, 2]. The operation of such systems is governed by a dual set of engineering challenges. Alongside the tasks of ensuring the requisite accuracy and speed in the positioning and movement of the working organs (end-effectors) and improving overall quality indicators, there is the pressing contemporary requirement of mitigating the energy consumption necessary for the execution of these movements [3-5].

Addressing these multiple objectives—precision, velocity, and energy efficiency—can be achieved most effectively through the systematic development and creation of more precise and computationally efficient control algorithms designed specifically for manipulation robotic systems [6]. The analytical determination of the system's kinematic model, as presented in this work, provides a critical computational advantage. An analytical, closed-form solution minimizes the processing delay required for state estimation within each cycle of the control loop, allowing for faster response times, higher accuracy, and consequently, a lower overall computational energy demand.

Results and Discussion

The core methodology employed in this research relies on the matrix approach for creating algorithms to calculate the kinematic parameters of multi-link manipulator robotic systems. The efficiency and expediency of this approach stem from the systematic application of sequential coordinate transformation when moving from one link in the chain to the next.

This transition process is defined by a sequential set of transformation matrices, each of which is associated with the coordinate system of a corresponding link in the mechanism's kinematic chain. Each coordinate system is formulated using the renowned (D-H) method. This method involves successive construction of coordinate systems and the formation of the corresponding homogeneous transformation matrices.

By utilizing these transformation matrices, the relationship between the coordinates P_i of any arbitrary point P on the i-th link of the manipulation robotic system can be established within the reference system that is movably connected to the adjacent $(i-1)$-th link. This fundamental algebraic relationship is defined by:

$$\bar{P}_{i-1} = {}^{i-1}A_i \cdot \bar{P}_i, \quad (1)$$

where ${}^{i-1}A_i$ denotes the elementary transformation matrix for adjacent coordinate systems, and $\bar{P}_{i-1}$ and $\bar{P}_i$ are the position vectors of the point P in the adjacent frames.

This formulation allows the position of every link to be described accurately relative to the position of its predecessor link in the system.

The elementary matrix ${}^{i-1}A_i$ represents a compound transformation and is defined as the product of fundamental rotational and translational transformations, incorporating the four geometric parameters of the kinematic chain: a_i, α_i, d_i, and Θ_i:

$${}^{i-1}A_i = T_{Z,d_i} \cdot T_{Z,\theta_i} \cdot T_{X,a_i} \cdot T_{X,\alpha_i}. \quad (2)$$

The matrix of the point's coordinates P in the $(i-1)$-th reference frame associated with the $(i\text{-}1)$-th link can be represented in homogeneous coordinates as $P_{i-1} = (X_{i-1}, Y_{i-1}, Z_{i-1}, 1)^T$. Similarly, the coordinates in the i-th frame are $P_i = (X_i, Y_i, Z_i, 1)^T$.

The matrix, ${}^0T_i$, which determines the pose (position and orientation) of the i-th coordinate system relative to the base (absolute) frame, is derived as the product of the sequence of individual homogeneous transformation matrices:

$$ {}^0T_i = {}^0A_1 \cdot {}^1A_2 \cdot \ldots \cdot {}^{i-1}A_i = \prod_{i=1}^{n} {}^{i-1}A_i = $$
$$ = \begin{bmatrix} X_i & Y_i & Z_i & P_i \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^0R_i & {}^0P_i \\ 0 & 1 \end{bmatrix}, \; i = 1,2,\ldots,n. \quad (3)$$

In this structure, ${}^0R_i$ is the rotation matrix defining the orientation of the i-th coordinate system relative to the base frame, and ${}^0P_i$ is the position vector connecting the origin of the base frame to the origin of the i-th coordinate system. The elegance of this algebraic structure, achieved through homogeneous coordinates, is that it allows complex, multi-stage rotations and translations to be resolved efficiently through simple matrix multiplication.

The forward kinematics problem itself consists precisely in defining the final position and orientation of the end-effector (working organ) of the robotic manipulation system with respect to the absolute coordinate system. This definition yields the final homogeneous transformation matrix T:

$$ T = {}^0A_1 \cdot {}^1A_2 \cdot \ldots \cdot {}^{n-1}A_n = $$
$$ = \prod_{i=1}^{n} {}^{i-1}A_i, \; i = 1,2,\ldots,n, \quad (4)$$

where n is the total number of links in the kinematic chain of the manipulator.

The specified movement laws for the robot's links can therefore be determined by calculating this specific product of homogeneous transformation matrices as the system transitions from the movable coordinate systems of the links to the absolute base frame. The resulting sequence of products forms the analytical algorithm for the calculation.

The analytical investigations were performed using an elementary model of a six-degree-of-freedom system, described as a three-link manipulator mechanism, with verification calculations carried out using the MathCAD software package.

The transformation matrices from the first coordinate system up to the sixth link (A_1 through A_6) are explicitly defined below, where c_i and s_i denote the cosine and sine of the respective rotation angles between the manipulator links:

$$ A_1 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$
$$ A_3 = \begin{bmatrix} c_3 & 0 & s_3 & a_3c_3 \\ s_3 & 0 & -c_3 & a_3s_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$
$$ A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (5) $$

The total homogeneous transformation matrix, T_6, is calculated as the sequential product of these individual link transformations, which can be factored for convenience:

$$ T_6 = A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5 \cdot A_6 = A_{13} \cdot A_{46}, \quad (6) $$

where $A_{13} = A_1 \cdot A_2 \cdot A_3$, $A_{46} = A_4 \cdot A_5 \cdot A_6$.

The expanded form of the matrix T_6 is structured as follows, containing the orientation components ($x_{6x}\ to\ z_{6z}$) and the position vector components (P_x, P_y, P_z):

$$T_6 = \begin{bmatrix} x_{6x} & y_{6x} & z_{6x} & p_x \\ x_{6y} & y_{6y} & z_{6y} & p_y \\ x_{6z} & y_{6z} & z_{6z} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (7)$$

The components of this matrix, representing the resulting orientation, are derived through algebraic expansion:

$x_{6x} = c_1(c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5s_6) - s_1(s_4c_5c_6 + c_4s_6),$
$x_{6y} = s_1(c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6) - c_1(s_4c_5c_6 + c_4s_6),$
$x_{6z} = -s_{23}(c_4c_5c_6 - s_4s_6) - c_{23}s_4c_6,$
$y_{6x} = c_1(-c_{23}(c_4c_5s_6 + s_4c_6) + s_{23}s_5s_6) - s_1(-s_4c_5c_6 + c_4c_6),$
$y_{6y} = s_1(-c_{23}(c_4c_5s_6 + s_4c_6) + s_{23}s_5s_6) + c_1(-s_4c_5c_6 + c_4c_6),$
$y_{6z} = s_{23}(c_4c_5c_6 + s_4c_6) + c_{23}s_5c_6,$
$z_{6x} = c_1(c_{23}c_4s_5 + s_{23}c_5) - s_1s_4s_5,$
$z_{6y} = s_1(c_{23}c_4s_5 + s_{23}c_5) + c_1s_4s_5,$
$z_{6z} = -s_{23}c_4c_5 + c_{23}c_5.$

Accordingly, the end-effector coordinates (P_x, P_y, P_z) are determined through the final position components of the transformation matrix, yielding the complete analytical:

$P_x = c_1(d_6(c_{23}c_4s_5 - s_{23}c_5) + s_{23}d_4 + a_3c_{23} + a_2c_2) - -s_1(d_6s_4s_5 + d_2),$
$P_y = s_1(d_6(c_{23}c_4s_5 + s_{23}c_5) + s_{23}d_4 + a_3c_{23} + a_2c_2) + c_1(d_6s_4s_5 + d_4),$ (8)
$P_z = d_6(c_{23}c_5 + s_{23}c_4s_5) + c_{23}d_4 + a_3c_{23} - a_2s_2.$

The derivation of these closed-form equations is crucial, as it validates the algebraic model for the manipulator's geometry. Such deterministic solutions are fundamental to developing the high-precision control algorithms targeted in the introductory section, as they eliminate the computational burden and potential for accumulated error associated with iterative solutions.

Conclusions

The work successfully employed analytical research to present a comprehensive algorithm for solving the forward kinematics problem of a robotic manipulation system. The output explicitly defines the coordinates and the orientation of the end-effector with respect to the absolute base coordinate system.

Verification of the algorithm, applied specifically to the three-link manipulator model, confirmed the accuracy of the software calculations performed. The key benefit of these analytical results is that they substantially simplify and accelerate the process of obtaining necessary kinematic data.

The demonstrated correctness and computational efficiency of this analytical model provide a solid foundation for the subsequent application of these results in crucial areas of robotics engineering, specifically for dynamic analysis and the creation of sophisticated control mechanisms. By establishing the accurate relationship between joint variables and end-effector pose, this work enables the accurate calculation of dynamic properties, such as necessary joint torques, which are essential for implementing advanced, energy-efficient control strategies for the manipulator.

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