DEVELOPMENT OF COMPUTATIONAL TECHNIQUES TO SOLVE PROBLEMS EXTERNAL AND INTERNAL BALLISTICS

РАЗВИТИЕ ВЫЧИСЛИТЕЛЬНЫХ МЕТОДОВ ДЛЯ РЕШЕНИЯ ЗАДАЧ ВНЕШНЕЙ И ВНУТРЕННЕЙ БАЛЛИСТИКИ

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Abstract: *The paper is dedicated to the improvement of existing and development of new mathematical methods for ballistics. We consider the interpolation of air drag function by using parabolic spline. Used spline does not require any additional conditions and is stable. The efficiency of the application of parabolic spline for different drag functions is shown. The spline approximation demonstates good accuracy, saves the behavior of drag functions and can be used in models of external ballistics. The Dirichlet boundary value problem for the unsteady-state equation of convection-diffusion-reaction with the prevailing convection are presented too. For differencing is used the parabolic spline that is continuous together with its first-order derivative which does not demand any additional conditions for its construction. Search for solution of the task in the form of a spline is applied to time-discretizable equation of convection-diffusion-reaction. For the received of difference scheme its monotonicity is proved. Theoretical researches and results of numerous experiments show that the offered scheme for the equation of convection-diffusion-reaction allows to solve boundary value problem for the wide range of values of the equation coefficients. This applies especially cases when convection far exceeds diffusion. The monotonic scheme provides stability of the computational solution, and application of a parabolic spline for its construction allows to reproduce the solution in the form of continuous function for the timepoints determined by discretization. There are given examples of calculations for a case of domination of the convective term over diffusion one.*

***KEYWORDS**: BALLISTICS, AIR DRAG, INTERPOLATION, PARABOLIC SPLINE, EQUATION OF CONVECTION-DIFFUSION-REACTION, MONOTONIC DIFFERENCE SCHEME*

1. Introduction

One of the possible directions to increase the efficiency of the software using in the ballistic calculations is improving existing and creating new mathematical methods. The paper considers the use of designed parabolic splines to improve the accuracy and reliability of the ballistics problems solution.

Examples of the parabolic spline using for interpolation the grid functions of air resistance, and the using of spline for the creation of monotonic difference schemes for non-stationary second-order equation containing the first derivative (convection-diffusion-reaction equation) are given. It is assumed that the convective term of the equation is significantly dominate diffusion.

The main factor influencing on the ballistic trajectory characteristics is air resistance. Typically, the air resistance laws are presented in tabular form, functions and require subsequent interpolation. We propose to use a parabolic spline for this purpose [1].

It is known that the vibrational motion of mortar shell fire at the plane described by the equation of parabolic type comprising first and second derivatives of the solution function. Powder burning processes are also described by such equations. These processes are sensitive to the input data and the coefficients of the equation values, so it is necessary to develop the stable numerical methods for solving boundary value problems for the convection-diffusion-reaction equations. In this work we propose to use a parabolic spline for the construction of monotonic difference scheme to provide stable algorithms for solving equations for large values of Peclet numbers.

2. Parabolic interpolation spline

We will use parabolic spline which is determined as: function $S_{n,\nu}(x)$ called a spline of n degree and ν defect (n, ν – integer numbers, $0 \le \nu \le n-1$) with nodes on the grid Δ , if:

- for each segment $[x_i, x_{i+1}]$ function $S_{n,\nu}(x)$ is a polynomial of n degree;
- $S_{n,\nu}(x) \in C^{n-\nu}[a,b]$, where $C^{n-\nu}[a,b]$ – sets of $n-\nu$ continuous differentiable functions on segment $[a,b]$.

Let two breakdown points be given on the interval $[a,b]$:

$$\Delta_x : a = x_0 < x_1 < \ldots < x_{N+1} = b,$$

$$\Delta_\tau : x_0 = \tau_0 < \tau_1 < \ldots < \tau_N = x_{N+1},$$

where $x_{i-1} < \tau_{i-1} < x_i$, $i = 1,2,\ldots,N$, $N \ge 2$.

In the nodes of the grid Δ_x a function $f_i = f(x_i)$, $i = 1,2,\ldots,N+1$ are set.

The points Δ_x are called nodes of interpolation, and τ_i $(i = 1,2,\ldots,N)$ are spline nodes.

We construct a parabolic spline of defect 1 on the interval $[a,b]$, which satisfies the conditions

(1) $$S(x_i) = f(x_i),\ i = 1,2,\ldots,N+1.$$

The value of the function at the nodes of the spline $\tau_1, \tau_2, \ldots \tau_{N-1}$ denote φ_i, $i = 1,2,\ldots,N-1.$

The statement [1]. Interpolation parabolic spline $S(x)$ ($\nu = 1, n = 2$) for a given breakdown Δ_x, Δ_τ of interval $[a,b]$, which satisfies the conditions (1) exists and is unique.

Proof. To construct the spline we use the Lagrange interpolation polynomial on each segments $[\tau_{i-1}, \tau_i]$, $i = 1,2,\ldots,N$

$$S(x) = f_0 \frac{(x-x_1)(x-\tau_1)}{(x_0-x_1)(x_0-\tau_1)} + f_1 \frac{(x-x_0)(x-\tau_1)}{(x_1-x_0)(x_1-\tau_1)} + \\ + \varphi_1 \frac{(x-x_0)(x-x_1)}{(\tau_1-x_0)(\tau_1-x_1)}$$

for $x \in [\tau_0, \tau_1]$,

$$S(x)=\varphi_{i-1}\frac{(x-x_i)(x-\tau_i)}{(\tau_{i-1}-x_i)(\tau_{i-1}-\tau_i)}+f_i\frac{(x-\tau_{i-1})(x-\tau_i)}{(x_i-\tau_{i-1})(x_i-\tau_i)}+$$
$$+\varphi_i\frac{(x-x_i)(x-\tau_{i-1})}{(\tau_i-x_i)(\tau_i-\tau_{i-1})}$$

for $x\in[\tau_{i-1},\tau_i],\ i=2,...,N-1$,

$$S(x)=\varphi_{N-1}\frac{(x-x_N)(x-x_{N+1})}{(\tau_{N-1}-x_N)(\tau_{N-1}-x_{N+1})}+$$
$$+f_N\frac{(x-\tau_{N-1})(x-x_{N+1})}{(x_N-\tau_{N-1})(x_N-x_{N+1})}+$$
$$+f_{N+1}\frac{(x-\tau_{N-1})(x-x_N)}{(x_{N+1}-\tau_{N-1})(x_{N+1}-x_N)}$$

for $x\in[\tau_{N-1},\tau_N]$.

Let grids Δ_x,Δ_τ be uniform with a step h and put $\tau_i=\frac{1}{2}(x_{i-1}+x_i)=x_{i-1/2}$.

Consider $S'(x)$ for $x\in[\tau_0,\tau_1]$.

We have

$$S'(x)=\frac{2}{3h^2}f_0[(x-x_1)+(x-\tau_1)]-$$
$$-\frac{2}{h^2}f_1[(x-x_0)+(x-\tau_1)]+\frac{4}{3h^2}\varphi_1[(x-x_0)+(x-x_1)].$$

Derivative at point τ_1-0 written as

$$S'(\tau_1-0)=\frac{1}{3h}f_0-\frac{3}{h}f_1+\frac{8}{3h}\varphi_1.$$

We write derivative for $x\in[\tau_1,\tau_2]$, as

$$S'(x)=\frac{2}{h^2}\varphi_1[(x-x_2)+(x-\tau_2)]-$$
$$-\frac{4}{h^2}f_2[(x-\tau_1)+(x-\tau_2)]+\frac{2}{h^2}\varphi_2[(x-x_2)+(x-\tau_1)].$$

Derivative at point τ_1+0 is written as

$$S'(\tau_1+0)=-\frac{3}{h}\varphi_1+\frac{4}{h}f_2-\frac{1}{h}\varphi_2.$$

Then, from the condition on the first derivative continuity at a point τ_1

$$S'(\tau_1-0)=S'(\tau_1+0),$$

we shall have

$$\frac{1}{3h}f_0-\frac{3}{h}f_1+\frac{8}{3h}\varphi_1=-\frac{3}{h}\varphi_1+\frac{4}{h}f_2-\frac{1}{h}\varphi_2$$

or

(2) $$\frac{17}{3}\varphi_1+\varphi_2=-\frac{1}{3}f_0+3f_1+4f_2.$$

We write the conditions $S'(\tau_i-0)=S'(\tau_i+0)$ for $i=2,...,N-1$. We shall find $S'(x)$ for $x\in[\tau_{i-1},\tau_i]$:

$$S'(x)=\frac{2}{3h^2}\varphi_{i-1}[(x-x_i)+(x-\tau_i)]-$$
$$-\frac{4}{h^2}f_i[(x-\tau_{i-1})+(x-\tau_i)]+\frac{2}{h^2}\varphi_i[(x-x_i)+(x-\tau_{i-1})].$$

We write derivative $S'(x)$ for $x\in[\tau_i,\tau_{i+1}]$

$$S'(x)=\frac{2}{3h^2}\varphi_i[(x-x_{i+1})+(x-\tau_{i+1})]-$$
$$-\frac{4}{h^2}f_{i+1}[(x-\tau_i)+(x-\tau_{i+1})]+\frac{2}{h^2}\varphi_{i+1}[(x-x_{i+1})+(x-\tau_i)].$$

From the condition of derivatives continuity we write

(3) $$\varphi_{i-1}+6\varphi_i+\varphi_{i+1}=4(f_i+f_{i+1}),\ i=2,...,N-1.$$

Similarly, using the condition of continuity $S'(\tau_{N-1}-0)=S'(\tau_{N-1}+0)$ we shall have

(4) $$\varphi_{N-2}+\frac{17}{3}\varphi_{N-1}=4f_{N-1}+3f_N-\frac{1}{3}f_{N+1}.$$

By combining expressions (2), (3) and (4), we get a system of linear algebraic equations with three-diagonal matrix:

$$\frac{17}{3}\varphi_1+\varphi_2=-\frac{1}{3}f_0-3f_1+4f_2,$$

(5) $$\varphi_{i-1}+6\varphi_i+\varphi_{i+1}=4(f_i+f_{i+1}),\ i=2,...,N-2,$$

$$\varphi_{N-2}+\frac{17}{3}\varphi_{N-1}=4f_N+3f_N-\frac{1}{3}f_{N+1}.$$

The system (5) has the diagonal advantage, which implies the existence of a single solution [2].

Parabolic spline can build using non-uniform grid. Construction, studies and research parabolic spline interpolation error on a non-uniform grid are given in [3].

3. *The use of parabolic spline interpolation for air resistance*

We shall apply a parabolic spline for 1943 year law interpolation. This law is given by the table of the function $C_x^{1943}(M)$ in points M_j [4]. Fig. 1 presents graphs of tabular function $C_x^{1943}(M)$ and spline that interpolates it and table presents spline coefficients for each segment $[\tau_{i-1},\tau_i],\ i=\overline{1,36}.$ Graph shows that spline maintains well the properties of its original function.

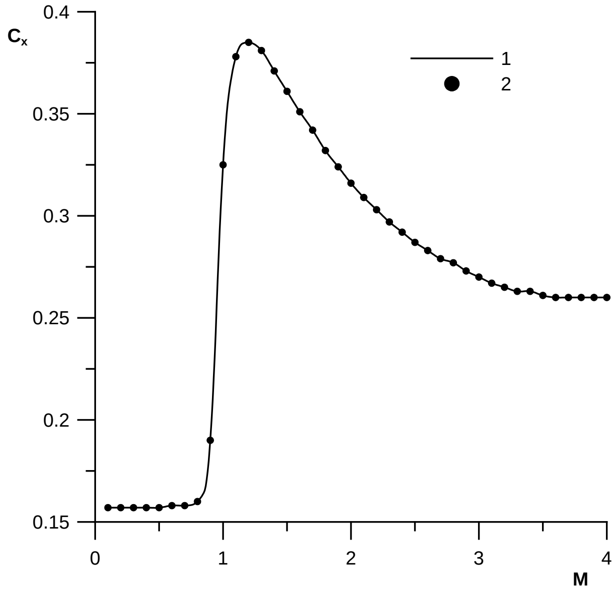

Fig. 1. The parabolic spline that interpolates air resistance 1943 year law, 1 – spline, 2 – value table function

It should be noted that the use of a cubic spline interpolation for table functions that are areas of rapid growth function may lead to its non-physical oscillations.

Fig. 2 shows a fragment of parabolic and cubic graphs [5] splines used for the 1943 year law interpolation. This example demonstrates one of the drawbacks of a cubic spline. Namely, for the step-like functions cubic spline function has variations in the vicinity of a sharp rising. This phenomenon is called the Gibbs effect [5]. It is evident that parabolic spline does not such fluctuations.

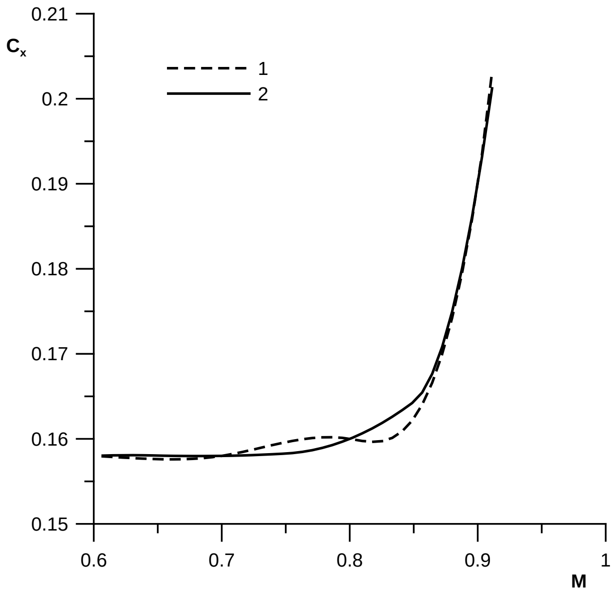

Fig. 2. Comparison of cubic and parabolic splines interpolations, 1 – cubic spline, 2 – parabolic spline

Thus, polynomials, that are components of the spline, have low degree, are easily implemented algorithmically and are continuous functions with the first derivative at all points of the segment interpolation.

4. Spline monotonic difference scheme for the of convection-diffusion-reaction equation

We consider the non-stationary boundary problem for the of convection-diffusion-reaction equation

(6) $$\frac{\partial u(x,t)}{\partial t} = D(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} - V(x,t)\frac{\partial u(x,t)}{\partial x} + A(x,t)u(x,t) + f(x,t),\ x \in (0,L),\ t > 0,$$

(7) $$u(0,t) = U_0(t),$$

(8) $$u(L,t) = U_L(t),$$

(9) $$u(x,0) = g(x),$$

where $V(x,t) \neq 0,\ 0 < D(x,t) << 1.$

We introduce a uniform grid by the time variable

$$\Delta_t : t_{k+1} = t_k + \rho,\ k = 0,1,2,...,t_0 = 0,\ \rho = const.$$

We write the problem as

(10) $$D(x,t_{k+1})\frac{d^2 u(x,t_{k+1})}{dx^2} - V(x,t_{k+1})\frac{du(x,t_{k+1})}{dx} - \frac{1}{\rho}u(x,t_{k+1}) + A(x,t_{k+1})u(x,t_{k+1}) = -f(x,t_{k+1}) - \frac{1}{\rho}u(x,t_k),$$

(11) $$u(0,t_{k+1}) = U_0(t_{k+1}),$$

(12) $$u(L,t_{k+1}) = U_L(t_{k+1}),$$

$$u(x,t_0) = g(x).$$

Let two breakdown Δ_x and Δ_τ be given on the interval $[0,L]$:

(13) a) $\Delta_x :\ 0 = x_0 < x_1 < ... < x_N = L,$

b) $\Delta_\tau :\ 0 = \tau_0 < \tau_1 < ... < \tau_{N-1} = L,$

where $x_i < \tau_i < x_{i+1},\ i = \overline{1, N-2}$.

Let C_i and φ_i be some grid values and functions, respectively, grids (13a) and (13b), and $\varphi_0 = C_0,\ \varphi_{N-1} = C_N$. We find a solution in the form of parabolic spline. For this we write the piecewise quadratic function $C(x)$ in $k+1$ moment of time, $x \in [\tau_i, \tau_{i+1}]$, $i = \overline{0, N-2}$, and find the first and second derivatives of the function on each of the segments, substitute them with the same function in equation (10). On transformation of the expressions derived from the conditions of continuity of the first-order derivatives of function, we receive system of the equations for internal nodes of the net domain at a point in time t_{k+1} [6, 7]:

(14) $$\alpha\varphi_{\tau_{i-1}}^{k+1} - \gamma\varphi_{\tau_i}^{k+1} + \beta\varphi_{\tau_{i+1}}^{k+1} = -\frac{f_{x_i}^{k+1} + f_{x_{i+1}}^{k+1}}{2} - \frac{u_{x_{i+1}}^k + u_{x_i}^k}{2\rho},\quad i = \overline{1, N-2},$$

where

$$\alpha = a + \frac{\mu^2}{2h^2}\left(\frac{1}{\rho} - A\right),$$

$$\gamma = a + b + \frac{2h^2 - \mu^2 - (h-\mu)^2}{2h^2}\left(\frac{1}{\rho} - A\right),$$

$$\beta = b - \frac{(h-\mu)^2}{2h^2}\left(\frac{1}{\rho} - A\right),$$

$$a = \frac{D}{h^2} + \frac{V}{h^2}\mu,\ b = \frac{D}{h^2} - \frac{V}{h}\left(1 - \frac{\mu}{h}\right).$$

The monotonicity of scheme is proved [2].

Results on the scheme approximation error are given in [7]. Note, that the presence of reaction in equation (6) does not affect the accuracy of the scheme.

5. Examples of calculations

Using developed software numerical calculations were carried out. The coefficients of the equation, initial and boundary conditions were set as follows: $L = 6$; $D = 0{,}0005$; $A = \pm 0{,}2$; $h = 0{,}001$; $u(0) = 1$; $u(L) = 0$; $u(x,0) = 0$; $V = 1$.

A comparison of numerical solution of the problem (6)–(9) with the exact solution when $A = 0$ is given in [5]. An expression $A \cdot u$ shall not affect the accuracy of numerical solution, so in this example we are interested only its effect on the quality characteristics of problem solution. Fig. 3, 4 shows a solutions of the problem as a "step" at different points in time.

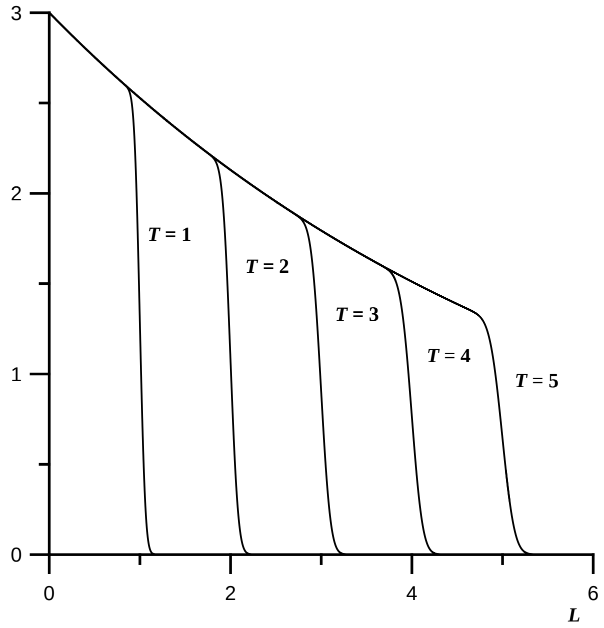

Fig. 3. The result of the calculation at a negative A

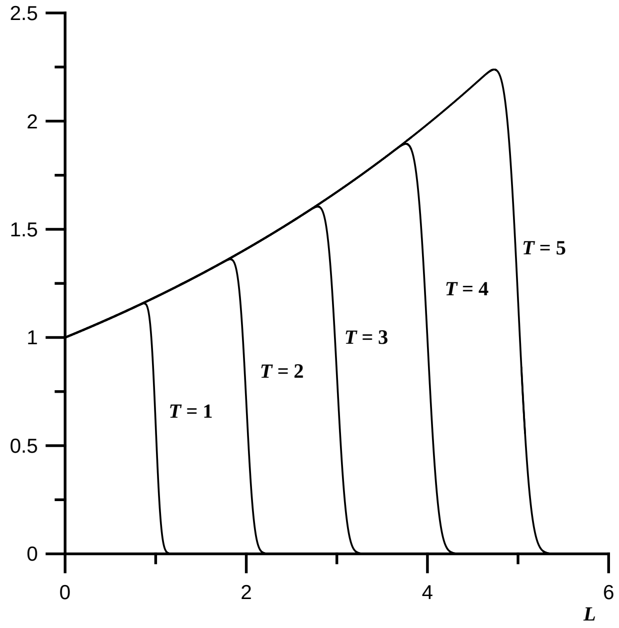

Fig. 4. The result of the calculation at a positive A

The numerical solution of the problem can be reproduced in a parabolic spline on the full interval of integration.

6. Conclusion

Thus, we can say that the parabolic spline is an appropriate tool for interpolation of tabular air resistance functions. It should also be noted that the functions presentation in the form of spline is convenient for use in numerical algorithms,

Theoretical studies and the results of numerical experiments show that proposed monotonice difference scheme for the convection-diffusion-reaction equation allows solving boundary value problems for a wide range of coefficients values. This is especially true when convection is much higher than diffusion. Monotonic scheme provides stability numerical solution, and using of parabolic spline allows us to construct a solution as a continuous function at every time moment.

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