SUFFICIENT CONDITION OF EXISTENCE WEYL FUNCTION FOR FRIEDRICHS' MODEL

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Abstract: *The paper proposes a mathematical model for describing the dynamics of multicomponent emulsions, based on ideas and methods of the kinetic theory of gases. The methodological basis of the proposed theory is the ideas and methods of the theory of integral kinetic equations.*
***KEYWORDS**: STURM-LIOUVILLE OPERATOR, FRIEDRICHS' MODEL, WEYL FUNCTION, CONTINUOUS SPECTRUM, BRANCHING OF RESOLVENT.*

1. Introduction

In classical case the Weil function is defined by connection between solutions the equations with equal boundary conditions. We suggest another approach in this paper. We will be able differentiate the solutions of the equations depending on the character of their properties of the analyticity. F

2. Statement of the problem

The problem is to give the notion of Weyl function for Friedrichs' model and which in the case of Sturm-Liouville operator coincides with classical notion of Weyl function.

The objective of this paper is:

To generalize the notion of Weil's function in case of nonselfadjoint Fridrihs' model, using the so-called branching of the resolvent. In the particular case of the Sturm-Liouville to get classical Weyl function (Sturm-Liouville previously included in the Friedrihs' model).

3. Branching resolvent function and Weil function

Let $H = L^2_\rho(0,\infty),\ \rho(\tau) > 0$. Suppose that interval $[0,\infty)$ coincides with continuous spectrum of some operator $T: H \to H,\quad \overline{D(T)} = H.$ Denote $T_\zeta = (T-\zeta)^{-1}$. Bilinear form of resolvent $(T_\zeta\varphi,\psi)$ is analytic function if $\zeta \notin [0,\infty)$.

Suppose that there exists linear space $\Phi \subset H,\quad \overline{\Phi} = H$ such that form $(T_\zeta\varphi,\psi),\quad \varphi,\psi \in \Phi$ admits analytic prolongation $(T_\zeta\varphi,\psi)_\pm$ over axis $(0,\infty)$. Suppose that $T:\Phi \to \Phi$ and that the multiplicity of continuous spectrum of T is $m=1$.

Denote by $h_\zeta \in H$ for $\zeta \in \Omega \setminus [0,\infty)$ analytic on ζ element in $H,\quad \Omega = \{\zeta : dist(\zeta,[0,\infty)) < \varepsilon\},\quad \varepsilon > 0.$ Let $a(\zeta), b(\zeta), r(\zeta)$ are some functionals in H and $B_\zeta : \Phi \to \Phi$ is some operator.

Definition 1. We say that the element $h_\zeta,\ \zeta \in \Omega \setminus [0,\infty)$ separates the branching of resolvent $T_\zeta,\ \zeta \in \Omega \setminus [0,\infty)$ if

$$T_\zeta\varphi = \left(\varphi, b(\overline{\zeta})\right)h_\zeta + B(\zeta)\varphi,\quad \varphi \in \Phi,\quad \zeta \in \Omega \setminus [0,\infty), \tag{2}$$

where the functions $\left(\varphi, \overline{b(\zeta)}\right)$ and $B(\zeta)\varphi,\ \varphi \in \Phi$ are analytic in Ω.

We say that scalar functions $M(\zeta),\ \zeta \in \Omega \setminus [0,\infty)$ separates the branching of h_ζ, if

$$(h_\zeta,\psi) = m(\zeta)(a(\zeta),\psi) + (r(\zeta),\psi),\quad \psi \in \Phi,\ \zeta \in \Omega \setminus [0,\infty), \tag{3}$$

where functions $(a(\zeta),\psi)$ and $(r(\zeta),\psi)$ are analytic in Ω. The fuction $m(\zeta)$ is called Weyl functions of the operator T.

In other words, the branching of resolvent T_ζ is given by the element h_ζ, and the branching of h_ζ is given by scalar function $m(\zeta)$.

We define $(T_\sigma\varphi,\psi)_\pm = \lim\limits_{\tau\to\pm0}(T_{\sigma+i\tau}\varphi,\psi),\quad \varphi,\psi \in \Phi$ and by analogy we define the elements $(h_\sigma)_\pm$ and the functions $m_\pm(\sigma), \sigma > 0.$

Further we suppose about T that there exist the elements φ, $\psi \in \Phi$, such that

$$(T_\sigma\varphi,\psi)_+ - (T_\sigma\varphi,\psi)_- \neq 0,\ \sigma > 0. \tag{4}$$

The sense of the functionals $a(\sigma)$, $b(\sigma)$ is given by following Lemma.

Lemma 1. The functionals a(σ),(b(σ)) in (2)-(3) are eigenfunctionals of operator T ,(T^*) corresponding to the point $\sigma \in (0,\infty)$ of continuous spectrum.

Some operator $T_1 \supset T$ is called extension of T if $D(T) \subset D(T_1)$ and $T_1\varphi = T\varphi,\ \varphi \in D(T)$.

4. Analysis Definition of the maximal operator $T_{\max}$

Definition 2. An extension $T_{\max} \supset T$ is called maximal operator for T :

1) if for every element $\varphi \in \Phi$ and every value $\sigma > 0$ there exists unique solution of the equation

$$(T_{\max} - \sigma) f_\sigma = \varphi,\ \sigma > 0,\ \varphi \in \Phi \qquad (5)$$

and if $f_\sigma \in \Phi$ too. We introduce the operator $T_{\max,\sigma} : \Phi \to \Phi$ as follows

$$T_{\max,\sigma}\varphi = (T_{\max} - \sigma)^{-1}\varphi = f_\sigma,\ \sigma > 0$$

2) if solution $T_{\max,\sigma}$ admits analytic prolongation f_ζ in the domain Ω such that $f_\zeta \in D(T_{\max})$ and $(T_{\max} - \zeta) f_\zeta = \varphi$, $\varphi \in \Phi$. Denote $T_{\max,\zeta}\varphi = f_\zeta$, then

$$(T_{\max} - \zeta) T_{\max,\zeta}\varphi = \varphi,\ \varphi \in \Phi,\ \zeta \in \Omega$$

Obviously, $T_{\max,\zeta} : \Phi \to \Phi$.

5. *Functional $c(\varphi)$ and branching T_ζ*

Definition 3. Denote by $c(\varphi)$ the functional on $D(T_{\max})$ defined by the condition

$$\varphi + c(\varphi) e \in D(T),\ \varphi \in D(T_{\max}). \qquad (8)$$

Theorem 1. Let $T_{\max} \supset T$ is maximal operator according to Def. 2. Then the resolvent of T admits the separating of branching

$$T_\zeta \varphi = (\varphi, b_{\bar\varsigma}) h_\varsigma + T_{\max,\zeta}\varphi,\ \varsigma \in \Omega \setminus [0,\infty), \qquad (9)$$

where

$$(\varphi, b_{\bar\varsigma}) = c(T_{\max,\varsigma}\varphi), \qquad (10)$$

and the element

$$h_\varsigma = e - T_\varsigma (T_{\max} - \varsigma) e \qquad (11)$$

is eigenvector of $T_{\max}$, namely

$$(T_{\max} - \varsigma) h_\varsigma = 0,\ c(h_\zeta) = -1, \qquad \varsigma \in \Omega \setminus [0,\infty). \qquad (12)$$

Remark 1. The Weyl function exits if operator $T_{\max}$ exists.

Let $H = L^2_\rho(0,\infty)$, $\rho(\tau) = \frac{1}{\pi}\sqrt{\tau}$. Denote by Φ, $\overline{\Phi} = H$ the subspace of the functions $\varphi(\tau)$, which admite analytic prolongation $\varphi(\zeta)$, $\zeta = \tau + i\mu$ in the domain Ω (see Def. 1). Denote by $S : H \to H$ the operator $S\varphi(\tau) = \tau\varphi(\tau), \tau > 0$ with maximal domain of definition $D(S)$. Let G is some Hilbert space and $V = A^* B$, where $A, B : H \to G$ are bounded integral operators. The operator

$$T = S + V,\quad V = A^* B,\quad D(T) = D(S),\quad R(A^*), R(B^*) \subset \Phi \qquad (13)$$

is called Friedrichs' model. Let us obtain an other definition of maximal operator $S_{\max}$ (see Def 2 and calculus in (12)- (14), where $T = S$). If $S_\zeta = (S - \zeta)^{-1}, \zeta \notin [0,\infty]$, then $S_\zeta \psi(\tau) = \frac{\psi(\tau)}{\tau - \zeta} = \psi(\zeta)\frac{1}{\tau - \zeta} + \frac{\psi(\tau) - \psi(\zeta)}{\tau - \zeta}$. This decomposition coincides with (15), therefore

$$S_{\max,\zeta}\psi(\tau) = \frac{\psi(\tau) - \psi(\zeta)}{\tau - \zeta}. \qquad (14)$$

Definition 4. Domain of definition:

$$D(S_{\max}) = \{\varphi \in H : \exists c(\varphi) : \tau\varphi(\tau) + c(\varphi) \in H\} \qquad (15)$$

and maximal operator

$$S_{\max}\varphi(\tau) = \tau\varphi(\tau) + c(\varphi), \quad \tau > 0. \qquad (16)$$

Definition 5. Let $(\cdot,1)$ is the functional on $D(S_{\max}) = D(S) + \left\{\frac{1}{\tau+1}\right\}$ defined by the relations

$$(\varphi,1) = \lim_{N\to\infty}(\varphi, 1_N),\ 1_N(x) = \chi_{[0,N]}(x), \qquad \varphi \in D(S), \qquad (17)$$

And $\left(\frac{1}{\tau+1}, 1\right) = -1.$ (18)

If $A: H \to H$ is bounded operator and $|(A^* f, 1)| \le C\|f\|$, $f \in H$, then we define the element $A \cdot 1$ by the relation $(f, A1) = (A^* f, 1)$, $f \in H$. (19)

Theorem 2. Denote $N(\zeta) = 1 + BS_{\max,\zeta} A^* \left(N_*(\zeta) = 1 + AS_{\max,\zeta} B^*\right)$ (see (22)). If the conditions

$$\left\|BS_{\max,\zeta} A^*\right\|_G < 1, \qquad \left\|AS_{\max,\zeta} B^*\right\|_G < 1, \qquad \zeta \in \Omega \tag{20}$$

hold then operator $T = S + A^* B\left(T^* = S + B^* A\right)$ has maximal operator

$$T_{\max} = S_{\max} + A^* B \qquad \left(\left(T^*\right)_{\max} = S_{\max} + B^* A\right) \tag{21}$$

The inverse operator

$$T_{\max,\sigma} = \left(T_{\max} - \sigma\right)^{-1} = S_{\max,\sigma} - S_{\max,\sigma} A^* N(\sigma)^{-1} BS_{\max,\sigma} \tag{22}$$

exists and the relations (6)-(7) hold.

Analitic prolongation of (22) gives:

$$T_{\max,\zeta}\varphi = S_{\max,\zeta}\varphi - S_{\max,\zeta} A^* N(\zeta)^{-1} BS_{\max,\zeta}. \tag{23}$$

If $T = S + A^* B$ and the operator $K(\zeta) = 1 + BS_\zeta A^*$, $\zeta \in [0, \infty)$ is invertible, then:

$$(T - \zeta)^{-1}\varphi = T_\zeta\varphi = S_\zeta\varphi - S_\zeta A^* K(\zeta)^{-1} BS_\zeta\varphi, \quad K(\zeta) = 1 + BS_\zeta A^*$$

The equation $\left(T_{\max} - \zeta\right)h_\zeta = 0$ (see (21), (16) and (23)) becomes (see Def 5)

$$(S - \zeta)h_\zeta + Vh_\zeta + c(h_\zeta) = 0, \ (T - \zeta)h_\zeta = -c(h_\zeta)\cdot 1, \ h_\zeta = -c(h_\zeta)T_\zeta 1.$$

In view of (12), we hold $c(h_\zeta) = -1$, so $h_\zeta = T_\zeta 1$. Therefore, the decomposition (9) of the resolvent for the operators T and T^* gives:

$$T_\zeta\varphi = \left(\varphi, b_{\bar\zeta}\right)T_\zeta 1 + T_{\max,\zeta}\varphi, \ \left(T^*\right)_{\bar\zeta}\psi = \left(\psi, a_{\bar\zeta}\right)\left(T^*\right)_{\bar\zeta} 1 + \left(T^*\right)_{\max,\bar\zeta}\psi. \tag{24}$$

Theorem 3. Let $T = S + A^* B$ (see(21)), then the conditions (20) are sufficient for existence Weyl function $m(\zeta)$ of the operator T (see Def.1). If the conditions (20) hold , then

$$m(\zeta) = \left(T_\zeta 1, 1\right), \quad \zeta \in \Omega \setminus [0, \infty). \tag{25}$$

Proof According to Theorem2 there exist operators $T_{\max}$ and $\left(T^*\right)_{\max}$. According to Def 2 (see(7)-(8)) we define operators $T_{\max}$, $\left(T^*\right)_{\max,\zeta}$. Using Theorem 1 and Definition 5, we obtain the relations (24), where the element $h_\zeta = T_\zeta 1$ separates the branching of $T_\zeta\varphi$. Finally,

$$\left(h_\zeta, \psi\right) = \left(T_\zeta 1, \psi\right) = \left(1, \left(T^*\right)_{\bar\zeta}\psi\right) = \overline{\left(\psi, a_{\bar\zeta}\right)}\left(1, \left(T^*\right)_{\bar\zeta} 1\right) + \left(1, \left(T^*\right)_{\max,\bar\zeta}\psi\right)$$

and in view of Definition 1, we have the relation (25), where $m(\zeta) = \left(1, \left(T^*\right)_{\bar\zeta} 1\right) = \left(T_\zeta 1, 1\right)$.

Theorem 3 is proved.

7. References

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